ISMRM Workshop on Data Sampling & Image Reconstruction

Session 5: Hard Core

Unifying Image Recon(struction)

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1. Image Reconstruction as Inverse Problem

Image reconstruction from discrete Fourier data in MRI is an ill-posed inverse problem.[1] The most important example is the reconstruction from undersampled multi-channel data based on coil sensitivities, *i.e.* SENSE-based parallel imaging.[2,3] Here, the forward model predicts multi-channel k-space samples $y_j(\vec{k}_n)$ on given k-space positions $\vec{k}_n$ for an estimate of the magnetization image ρ and given coil sensitivities c_j according to

$$y_j(\vec{k}_n) = \int d\vec{r}\; c_j(\vec{r})\rho(\vec{r})e^{-i2\pi\vec{k}_n\cdot\vec{r}} \quad .$$

Even more complicated signal models can be used, *e.g.* by including terms for relaxation or off-resonance effects or even full Bloch equations.[4-6] In general, a solution of the inverse problem may be defined as the minimizer of the following least-squares optimization problem with regularization:

$$x_\alpha = argmin_x\{\|Ax-y\|_2^2 + R(x)\}$$

The regularization term $R(x)$ incorporates prior knowledge, *i.e.* information about structure of the unknown image. After discretization, a numerical optimization method can be applied to compute an approximation to this solution. This formulation is very generic and can be used for reconstruction in non-Cartesian MRI, parallel imaging, compressed sensing, and model-based approaches.[2-9] As an example, it is possible to apply this approach to autocalibrating parallel imaging. Here, in addition to the image also the coil sensitivities are considered unknowns and, consequently, the problem becomes a non-linear blind deconvolution problem.[10,11]

2. A Sampling Theory for Multi-Channel k-Space

The forward model can be split into a signal model which predicts ideal continuous k-space signals $f_j^\rho(\vec{k})$ and into a sampling operator which evaluates the k-space signals at discrete sample positions $\vec{k}_n$. Mathematically, the space of ideal continuous k-space signals can be identified as a certain reproducing kernel Hilbert space (RKHS) which is uniquely characterized by its kernel.[12] In parallel imaging it can be shown to be defined in terms of the sensitivities:[13]

$$K_{st}(\vec{k},\vec{l}) = \int_{FOV} d\vec{r}\; c_s(\vec{r})\overline{c_t(\vec{r})}e^{-2\pi i(\vec{k}-\vec{l})\cdot\vec{r}} \quad .$$

This kernel completely describes all correlations in k-space that are induced by the coil sensitivities and the field-of-view (FOV). Sampling in an RKHS can be formulated using approximation theory which provides an optimal reconstruction formula with error bounds.

3. The Relationship to k-Space Methods

GRAPPA is an alternative approach to parallel imaging that uses linear prediction of the missing samples from local neighborhoods in k-space.[14] A reformulation shows that it uses the same formula as known from approximation theory but with a different kernel: Instead of the ideal kernel derived from the coil sensitivities it uses an empirical estimate of the covariance function.[13] It is possible to translate the advantages of GRAPPA into the framework of inverse problems by deriving explicit sensitivity maps from a k-space reconstruction operator. This can be done with ESPIRiT, where sensitivity maps are computed using a point-wise eigenvalue analysis applied to an reconstruction operator derived from the estimated covariance matrix (or calibration matrix).[15] Exactly in those situations where GRAPPA appears to be more robust than SENSE, multiple sets of such maps emerge in the eigenvalue analysis which can then be used in an extended forward model to achieve similar robustness as GRAPPA.

4. Generalized Reconstruction Software

Depending on the forward model and the regularization terms, efficient optimization of the discretized problem may require the use of specific numerical algorithms. A very flexible approach is based on Alternating Directions Methods of Multipliers (ADMM), which splits the optimization problem into smaller problems that can often be solved efficiently using subroutines. This flexibility is exploited in the Berkeley Advanced Reconstruction Tools[16], where various forward operators can be combined with an arbitrary number of regularization terms to construct many advanced reconstruction methods.

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