1. **Purpose**

For a bivariate time series, `nag_tsa_cross_spectrum_bivar (g13cec)` calculates the cross amplitude spectrum and squared coherency, together with lower and upper bounds from the univariate and bivariate (cross) spectra.

2. **Specification**

```c
#include <nag.h>
#include <nagg13.h>

void nag_tsa_cross_spectrum_bivar(double xg[], double yg[], Complex xyg[],
                                        Integer ng, double stats[], double ca[], double calw[],
                                        double caup[], double *t, double sc[], double sclw[],
                                        double scup[], NagError *fail)
```

3. **Description**

Estimates of the cross amplitude spectrum $A(\omega)$ and squared coherency $W(\omega)$ are calculated for each frequency $\omega$ as

$$A(\omega) = |f_{xy}(\omega)| = \sqrt{cf(\omega)^2 + qf(\omega)^2}$$

and

$$W(\omega) = \frac{|f_{xy}(\omega)|^2}{f_{xx}(\omega)f_{yy}(\omega)}$$

where:

$cf(\omega)$ and $qf(\omega)$ are the co-spectrum and quadrature spectrum estimates between the series, i.e., the real and imaginary parts of the cross spectrum $f_{xy}(\omega)$ as obtained using `nag_tsa_spectrum_bivar (g13cdc)`. $f_{xx}(\omega)$ and $f_{yy}(\omega)$ are the univariate spectrum estimates for the two series as obtained using `nag_tsa_spectrum_univar (g13cbc)`.

The same type and amount of smoothing should be used for these estimates, and this is specified by the degrees of freedom and bandwidth values which are passed from the calls of `nag_tsa_spectrum_univar (g13cbc)`.

Upper and lower 95% confidence limits for the cross amplitude are given approximately by

$$A(\omega) \left[ 1 \pm \left( \frac{1.96}{\sqrt{d}} \right) \sqrt{W(\omega)^{-1} + 1} \right],$$

except that a negative lower limit is reset to 0.0, in which case the approximation is rather poor. The user is therefore particularly recommended to compare the coherency estimate $W(\omega)$ with the critical value $T$ derived from the upper 5% point of the $F$-distribution on $(2,d-2)$ degrees of freedom:

$$T = \frac{2F}{d - 2 + 2F}$$

where $d$ is the degrees of freedom associated with the univariate spectrum estimates. The value of $T$ is returned by the routine.

The hypothesis that the series are unrelated at frequency $\omega$, i.e., that both the true cross amplitude and coherency are zero, may be rejected at the 5% level if $W(\omega) > T$. Tests at two frequencies separated by more than the bandwidth may be taken to be independent.

The confidence limits on $A(\omega)$ are strictly appropriate only at frequencies for which the coherency is significant. The same applies to the confidence limits on $W(\omega)$ which are however calculated at all frequencies using the approximation that $\arctanh \left( \sqrt{W(\omega)} \right)$ is Normal with variance $1/d$. 

4. Parameters

\texttt{xg[ng]}

Input: the \texttt{ng} univariate spectral estimates, \( f_{xx}(\omega) \), for the \texttt{x} series.

\texttt{yg[ng]}

Input: the \texttt{ng} univariate spectral estimates, \( f_{yy}(\omega) \), for the \texttt{y} series.

\texttt{xyg[ng]}

Input: \( f_{xy}(\omega) \), the \texttt{ng} bivariate spectral estimates for the \texttt{x} and \texttt{y} series. The \texttt{x} series leads the \texttt{y} series.

\textbf{Note:} the two univariate and the bivariate spectra must each have been calculated using the same amount of smoothing. The frequency width and the shape of the window and the frequency division of the spectral estimates must be the same. The spectral estimates and statistics must also be unlogged.

\texttt{ng}

Input: the number of spectral estimates in each of the arrays \texttt{xg}, \texttt{yg} and \texttt{xyg}. It is also the number of cross amplitude spectral and squared coherency estimates.

Constraint: \( \texttt{ng} \geq 1 \).

\texttt{stats[4]}

Input: the 4 associated statistics for the univariate spectral estimates for the \texttt{x} and \texttt{y} series. \texttt{stats[0]} contains the degrees of freedom, \texttt{stats[1]} and \texttt{stats[2]} contain the lower and upper bound multiplying factors respectively and \texttt{stats[3]} contains the bandwidth.

Constraints:
\[
\begin{align*}
\texttt{stats[0]} &\geq 3.0, \\
0.0 &< \texttt{stats[1]} \leq 1.0, \\
\texttt{stats[2]} &\geq 1.0.
\end{align*}
\]

\texttt{ca[ng]}

Output: the \texttt{ng} cross amplitude spectral estimates \( \hat{A}(\omega) \) at each frequency of \( \omega \).

\texttt{calw[ng]}

Output: the \texttt{ng} lower bounds for the \texttt{ng} cross amplitude spectral estimates.

\texttt{caup[ng]}

Output: the \texttt{ng} upper bounds for the \texttt{ng} cross amplitude spectral estimates.

\texttt{t}

Output: the critical value for the significance of the squared coherency, \( T \).

\texttt{sc[ng]}

Output: the \texttt{ng} squared coherency estimates, \( \hat{W}(\omega) \) at each frequency \( \omega \).

\texttt{sclw[ng]}

Output: the \texttt{ng} lower bounds for the \texttt{ng} squared coherency estimates.

\texttt{scup[ng]}

Output: the \texttt{ng} upper bounds for the \texttt{ng} squared coherency estimates.

\texttt{fail}

The NAG error parameter, see the Essential Introduction to the NAG C Library.

5. Error Indications and Warnings

\texttt{NE_INT_ARG_LT}

On entry, \texttt{ng} must not be less than 1: \( \texttt{ng} = \langle \texttt{value} \rangle \).

\texttt{NE_REAL_ARG_LT}

On entry, \texttt{stats[0]} must not be less than 3.0: \( \texttt{stats[0]} = \langle \texttt{value} \rangle \).

On entry, \texttt{stats[2]} must not be less than 1.0: \( \texttt{stats[2]} = \langle \texttt{value} \rangle \).

\texttt{NE_REAL_ARG_LE}

On entry, \texttt{stats[1]} must not be less than or equal to 0.0: \( \texttt{stats[1]} = \langle \texttt{value} \rangle \).

\texttt{NE_REAL_ARG_GT}

On entry, \texttt{stats[1]} must not be greater than 1.0: \( \texttt{stats[1]} = \langle \texttt{value} \rangle \).
NE_BIVAR_SPECTRAL_ESTIM_ZERO
A bivariate spectral estimate is zero.
For this frequency the cross amplitude spectrum is set to zero, and the contributions to the
impulse response function and its standard error are set to zero.

NE_UNIVAR_SPECTRAL_ESTIM_NEG
A bivariate spectral estimate is negative.
For this frequency the cross amplitude spectrum is set to zero, and the contributions to the
impulse response function and its standard error are set to zero.

NE_UNIVAR_SPECTRAL_ESTIM_ZERO
A bivariate spectral estimate is zero.
For this frequency the cross amplitude spectrum is set to zero, and the contributions to the
impulse response function and its standard error are set to zero.

NE_SQUARED_FREQ_GT_ONE
A calculated value of the squared coherency exceeds one.
For this frequency the squared coherency is reset to one with the result that the cross
amplitude spectrum is zero and the contribution to the impulse response function at this
frequency is zero.

NE_ALLOC_FAIL
Memory allocation failed.

NE_INTERNAL_ERROR
An internal error has occurred in this function. Check the function call and any array sizes.
If the call is correct then please consult NAG for assistance.

6. Further Comments
The time taken by the routine is approximately proportional to \( ng \).

6.1. Accuracy
All computations are very stable and yield good accuracy.

6.2. References

7. See Also
None

8. Example
The example program reads the set of univariate spectrum statistics, the 2 univariate spectra
and the cross spectrum at a frequency division of \( \frac{2\pi}{20} \) for a pair of time series. It calls
nag_tsa_cross_spectrum_bivar to calculate the cross amplitude spectrum and squared coherency
and their bounds and prints the results.
8.1. Program Text

/* nag_tsa_cross_spectrum_bivar(g13cec) Example Program.  
* Mark 4, 1996.  
*/

#include <nag.h>
#include <stdio.h>
#include <nag_stdlib.h>
#include <naga02.h>
#include <nagg13.h>

#define L 80
#define KC 8*L
#define NGMAX KC
#define NXYMAX 300

main()
{
    double stats[4];
    double x[KC], y[KC];
    double pxy;
    double pw;
    double ca[NGMAX], calw[NGMAX], caup[NGMAX], sc[NGMAX], sclw[NGMAX],
    scup[NGMAX];
    double t;
    double *xg, *yg;
    Complex *xyg;

    Integer i, j, ng, is;
    Integer mw;
    Integer nxy;
    Integer kc=KC, l=L;

    Vprintf("g13cec Example Program Results\n");
    /* Skip heading in data file */
    Vscanf("%*[\n"]");
    Vscanf("%ld ", &nxy);
    if (nxy > 0 && nxy <= NXYMAX)
    {
        for (i = 1; i <= nxy; ++i)
            Vscanf("%lf ", &x[i - 1]);
        for (i = 1; i <= nxy; ++i)
            Vscanf("%lf ", &y[i - 1]);

        /* Set parameters for call to g13cbc and g13cdc  
        * with mean correction and 10 percent taper  
        */
        pxy = 0.1;
        /* Window shape parameter and zero covariance at lag 16 */
        pw = .5;
        mw = 16;
        /* Alignment shift of 3 */
        is = 3;

        /* Obtain univariate spectrum for the x and the y series */
        g13cbc(nxy, Nag_Mean, pxy, mw, pw, l, kc, Nag_Unlogged, x, &xg,
        &ng, stats, NAGERR_DEFAULT);
        g13cbc(nxy, Nag_Mean, pxy, mw, pw, l, kc, Nag_Unlogged, y, &yg,
        &ng, stats, NAGERR_DEFAULT);

        /* Obtain cross spectrum of the bivariate series */
g13 – Time Series Analysis

g13cdc(nxy, Nag_Mean, pxy, mw, is, pw, l, kc, x, y, &xyg, 
&ng, NAGERR_DEFAULT);

g13cec(xg, yg, xyg, ng, stats, ca, calw, caup, &t, sc, sclw, 
scup, NAGERR_DEFAULT);

Vprintf("\n");
Vprintf(" Cross amplitude spectrum\n\n");
Vprintf(" Lower Upper\n");
for (j = 1; j <= ng; ++j)
  Vprintf(" %5ld%10.4f%10.4f%10.4f
", 
        j - 1, ca[j - 1], calw[j - 1], caup[j - 1]);
Vprintf("\n");
Vprintf(" Squared coherency test statistic =%12.4f

", t);
Vprintf(" Squared coherency\n");
Vprintf(" Lower Upper\n");
for (j = 1; j <= ng; ++j)
  Vprintf(" %5ld%10.4f%10.4f%10.4f
", 
        j - 1, sc[j - 1], sclw[j - 1], scup[j - 1]);
}
NAG_FREE(xg);
NAG_FREE(yg);
NAG_FREE(xyg);
exit(EXIT_SUCCESS);

8.2. Program Data

g13cec Example Program Data

296
-0.109 0.000 0.178 0.339 0.373 0.441 0.461 0.348
0.127 -0.180 0.000 0.178 0.339 0.373 0.441 0.461
0.127 -0.180 0.000 0.178 0.339 0.373 0.441 0.461
-0.109 0.000 0.178 0.339 0.373 0.441 0.461 0.348
0.127 -0.180 0.000 0.178 0.339 0.373 0.441 0.461
0.127 -0.180 0.000 0.178 0.339 0.373 0.441 0.461
0.127 -0.180 0.000 0.178 0.339 0.373 0.441 0.461
-0.109 0.000 0.178 0.339 0.373 0.441 0.461 0.348
0.127 -0.180 0.000 0.178 0.339 0.373 0.441 0.461
0.127 -0.180 0.000 0.178 0.339 0.373 0.441 0.461
-0.109 0.000 0.178 0.339 0.373 0.441 0.461 0.348
0.127 -0.180 0.000 0.178 0.339 0.373 0.441 0.461
0.127 -0.180 0.000 0.178 0.339 0.373 0.441 0.461
0.127 -0.180 0.000 0.178 0.339 0.373 0.441 0.461
-0.109 0.000 0.178 0.339 0.373 0.441 0.461 0.348
0.127 -0.180 0.000 0.178 0.339 0.373 0.441 0.461
0.127 -0.180 0.000 0.178 0.339 0.373 0.441 0.461
0.127 -0.180 0.000 0.178 0.339 0.373 0.441 0.461
-0.109 0.000 0.178 0.339 0.373 0.441 0.461 0.348
0.127 -0.180 0.000 0.178 0.339 0.373 0.441 0.461
0.127 -0.180 0.000 0.178 0.339 0.373 0.441 0.461
0.127 -0.180 0.000 0.178 0.339 0.373 0.441 0.461
-0.109 0.000 0.178 0.339 0.373 0.441 0.461 0.348
0.127 -0.180 0.000 0.178 0.339 0.373 0.441 0.461
0.127 -0.180 0.000 0.178 0.339 0.373 0.441 0.461
0.127 -0.180 0.000 0.178 0.339 0.373 0.441 0.461
-0.109 0.000 0.178 0.339 0.373 0.441 0.461 0.348
0.127 -0.180 0.000 0.178 0.339 0.373 0.441 0.461
0.127 -0.180 0.000 0.178 0.339 0.373 0.441 0.461
0.127 -0.180 0.000 0.178 0.339 0.373 0.441 0.461
53.8 53.6 53.5 53.5 53.4 53.1 52.7 52.4 52.2 52.0 52.0 52.4 53.0 54.0 54.9 56.0
### Program Results

**g13cec Example Program Results**

Cross amplitude spectrum

<table>
<thead>
<tr>
<th>Value</th>
<th>Lower bound</th>
<th>Upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6.1563</td>
<td>3.6901</td>
</tr>
<tr>
<td>1</td>
<td>5.9415</td>
<td>3.5602</td>
</tr>
<tr>
<td>2</td>
<td>4.3034</td>
<td>2.5797</td>
</tr>
<tr>
<td>3</td>
<td>2.6086</td>
<td>1.5642</td>
</tr>
<tr>
<td>4</td>
<td>1.8249</td>
<td>1.0918</td>
</tr>
<tr>
<td>5</td>
<td>1.1865</td>
<td>0.7060</td>
</tr>
<tr>
<td>6</td>
<td>0.9616</td>
<td>0.5694</td>
</tr>
<tr>
<td>7</td>
<td>0.6893</td>
<td>0.4080</td>
</tr>
<tr>
<td>8</td>
<td>0.4629</td>
<td>0.2728</td>
</tr>
<tr>
<td>9</td>
<td>0.2398</td>
<td>0.1407</td>
</tr>
<tr>
<td>10</td>
<td>0.1199</td>
<td>0.0698</td>
</tr>
<tr>
<td>11</td>
<td>0.0769</td>
<td>0.0441</td>
</tr>
<tr>
<td>12</td>
<td>0.0463</td>
<td>0.0262</td>
</tr>
<tr>
<td>13</td>
<td>0.0301</td>
<td>0.0166</td>
</tr>
<tr>
<td>14</td>
<td>0.0158</td>
<td>0.0083</td>
</tr>
<tr>
<td>15</td>
<td>0.0089</td>
<td>0.0046</td>
</tr>
<tr>
<td>16</td>
<td>0.0050</td>
<td>0.0024</td>
</tr>
<tr>
<td>17</td>
<td>0.0016</td>
<td>0.0007</td>
</tr>
<tr>
<td>18</td>
<td>0.0004</td>
<td>0.0001</td>
</tr>
<tr>
<td>19</td>
<td>0.0004</td>
<td>0.0001</td>
</tr>
<tr>
<td>20</td>
<td>0.0003</td>
<td>0.0001</td>
</tr>
<tr>
<td>21</td>
<td>0.0003</td>
<td>0.0000</td>
</tr>
<tr>
<td>22</td>
<td>0.0004</td>
<td>0.0001</td>
</tr>
<tr>
<td>23</td>
<td>0.0003</td>
<td>0.0000</td>
</tr>
<tr>
<td>24</td>
<td>0.0002</td>
<td>0.0000</td>
</tr>
<tr>
<td>25</td>
<td>0.0004</td>
<td>0.0001</td>
</tr>
<tr>
<td>26</td>
<td>0.0003</td>
<td>0.0001</td>
</tr>
<tr>
<td>27</td>
<td>0.0001</td>
<td>0.0000</td>
</tr>
<tr>
<td>28</td>
<td>0.0002</td>
<td>0.0000</td>
</tr>
<tr>
<td>29</td>
<td>0.0003</td>
<td>0.0001</td>
</tr>
<tr>
<td>30</td>
<td>0.0004</td>
<td>0.0001</td>
</tr>
<tr>
<td>31</td>
<td>0.0002</td>
<td>0.0000</td>
</tr>
<tr>
<td>32</td>
<td>0.0001</td>
<td>0.0000</td>
</tr>
<tr>
<td>33</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>34</td>
<td>0.0001</td>
<td>0.0000</td>
</tr>
<tr>
<td>35</td>
<td>0.0002</td>
<td>0.0000</td>
</tr>
<tr>
<td>36</td>
<td>0.0002</td>
<td>0.0000</td>
</tr>
<tr>
<td>37</td>
<td>0.0001</td>
<td>0.0000</td>
</tr>
<tr>
<td>38</td>
<td>0.0001</td>
<td>0.0000</td>
</tr>
<tr>
<td>39</td>
<td>0.0001</td>
<td>0.0000</td>
</tr>
<tr>
<td>40</td>
<td>0.0001</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Squared coherency test statistic = 0.1926
### Squared coherency

<table>
<thead>
<tr>
<th>Value</th>
<th>Lower bound</th>
<th>Upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.9562</td>
<td>0.9783</td>
</tr>
<tr>
<td>1</td>
<td>0.9539</td>
<td>0.9772</td>
</tr>
<tr>
<td>2</td>
<td>0.9567</td>
<td>0.9786</td>
</tr>
<tr>
<td>3</td>
<td>0.9591</td>
<td>0.9798</td>
</tr>
<tr>
<td>4</td>
<td>0.9428</td>
<td>0.9716</td>
</tr>
<tr>
<td>5</td>
<td>0.9048</td>
<td>0.9523</td>
</tr>
<tr>
<td>6</td>
<td>0.8738</td>
<td>0.9362</td>
</tr>
<tr>
<td>7</td>
<td>0.8715</td>
<td>0.9350</td>
</tr>
<tr>
<td>8</td>
<td>0.8453</td>
<td>0.9212</td>
</tr>
<tr>
<td>9</td>
<td>0.8198</td>
<td>0.9075</td>
</tr>
<tr>
<td>10</td>
<td>0.7770</td>
<td>0.8841</td>
</tr>
<tr>
<td>11</td>
<td>0.7033</td>
<td>0.8422</td>
</tr>
<tr>
<td>12</td>
<td>0.6541</td>
<td>0.8131</td>
</tr>
<tr>
<td>13</td>
<td>0.5702</td>
<td>0.7609</td>
</tr>
<tr>
<td>14</td>
<td>0.4474</td>
<td>0.6776</td>
</tr>
<tr>
<td>15</td>
<td>0.3945</td>
<td>0.6385</td>
</tr>
<tr>
<td>16</td>
<td>0.3260</td>
<td>0.5844</td>
</tr>
<tr>
<td>17</td>
<td>0.1887</td>
<td>0.4580</td>
</tr>
<tr>
<td>18</td>
<td>0.0390</td>
<td>0.2566</td>
</tr>
<tr>
<td>19</td>
<td>0.0618</td>
<td>0.2975</td>
</tr>
<tr>
<td>20</td>
<td>0.0524</td>
<td>0.2815</td>
</tr>
<tr>
<td>21</td>
<td>0.0428</td>
<td>0.2639</td>
</tr>
<tr>
<td>22</td>
<td>0.0578</td>
<td>0.2907</td>
</tr>
<tr>
<td>23</td>
<td>0.0282</td>
<td>0.2337</td>
</tr>
<tr>
<td>24</td>
<td>0.0220</td>
<td>0.2188</td>
</tr>
<tr>
<td>25</td>
<td>0.0807</td>
<td>0.3268</td>
</tr>
<tr>
<td>26</td>
<td>0.0754</td>
<td>0.3189</td>
</tr>
<tr>
<td>27</td>
<td>0.0167</td>
<td>0.2047</td>
</tr>
<tr>
<td>28</td>
<td>0.0331</td>
<td>0.2445</td>
</tr>
<tr>
<td>29</td>
<td>0.0777</td>
<td>0.3223</td>
</tr>
<tr>
<td>30</td>
<td>0.1414</td>
<td>0.4059</td>
</tr>
<tr>
<td>31</td>
<td>0.0739</td>
<td>0.3166</td>
</tr>
<tr>
<td>32</td>
<td>0.0107</td>
<td>0.1861</td>
</tr>
<tr>
<td>33</td>
<td>0.0008</td>
<td>0.1358</td>
</tr>
<tr>
<td>34</td>
<td>0.0488</td>
<td>0.2750</td>
</tr>
<tr>
<td>35</td>
<td>0.0760</td>
<td>0.3198</td>
</tr>
<tr>
<td>36</td>
<td>0.0738</td>
<td>0.3165</td>
</tr>
<tr>
<td>37</td>
<td>0.0719</td>
<td>0.3135</td>
</tr>
<tr>
<td>38</td>
<td>0.0438</td>
<td>0.2657</td>
</tr>
<tr>
<td>39</td>
<td>0.0168</td>
<td>0.2049</td>
</tr>
<tr>
<td>40</td>
<td>0.0183</td>
<td>0.2091</td>
</tr>
</tbody>
</table>