nag_opt_lp (e04mfc)

1. Purpose

nag_opt_lp solves general linear programming problems. It is not intended for large sparse problems.

2. Specification

```c
#include <nag.h>
#include <nage04.h>

void nag_opt_lp(Integer n, Integer nclin, double a[], Integer tda, double bl[], double bu[], double cvec[], double x[], double *objf, Nag_E04_Opt *options, Nag_Comm *comm, NagError *fail)
```

3. Description

nag_opt_lp is designed to solve linear programming (LP) problems of the form

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad l \leq \begin{bmatrix} x \\ Ax \end{bmatrix} \leq u,
\end{align*}
\]

where \( c \) is an \( n \) element vector and \( A \) is an \( m_{\text{lin}} \) by \( n \) matrix.

The routine allows the linear objective function to be omitted in which case a feasible point (FP) for the set of constraints is sought.

The constraints involving \( A \) are called the general constraints. Note that upper and lower bounds are specified for all the variables and for all the general constraints. An equality constraint can be specified by setting \( l_i = u_i \). If certain bounds are not present, the associated elements of \( l \) or \( u \) can be set to special values that will be treated as \( -\infty \) or \( +\infty \). (See the description of the optional parameter inf_bound in Section 8.2).

The user must supply an initial estimate of the solution.

Details about the algorithm are described in Section 7, but it is not necessary to read this more advanced section before using nag_opt_lp.

4. Parameters

\( n \)

Input: \( n \), the number of variables.
Constraint: \( n > 0 \).

\( n_{\text{clin}} \)

Input: \( m_{\text{lin}} \), the number of general linear constraints.
Constraint: \( n_{\text{clin}} \geq 0 \).

\( a[n_{\text{clin}}][tda] \)

Input: the \( i \)th row of \( a \) must contain the coefficients of the \( i \)th general linear constraint (the \( i \)th row of \( A \)), for \( i = 1, 2, \ldots, m_{\text{lin}} \).
If \( n_{\text{clin}} = 0 \) then the array \( a \) is not referenced.

\( tda \)

Input: the second dimension of the array \( a \) as declared in the function from which nag_opt_lp is called.
Constraint: \( tda \geq n \) if \( n_{\text{clin}} > 0 \).

\( bl[n+n_{\text{clin}}] \)

\( bu[n+n_{\text{clin}}] \)

Input: \( bl \) must contain the lower bounds and \( bu \) the upper bounds, for all the constraints in the following order. The first \( n \) elements of each array must contain the bounds on the
variables, and the next \( m_{\text{lin}} \) elements the bounds for the general linear constraints (if any).

To specify a non-existent lower bound (i.e., \( l_j = -\infty \)), set \( bl[j] \leq -\text{inf\_bound} \), and to specify a non-existent upper bound (i.e., \( u_j = +\infty \)), set \( bu[j] \geq \text{inf\_bound} \); here \( \text{inf\_bound} \) is the value of the optional parameter \texttt{options.inf\_bound}, whose default value is \( 10^{20} \) (see Section 8.2).

To specify the \( j \)th constraint as an equality, set \( bl[j] = bu[j] = \beta \), say, where \( |\beta| < \text{inf\_bound} \).

Constraint: \( bl[j] \leq bu[j] \), for \( j = 0, 1, \ldots, n + m_{\text{lin}} - 1 \).

cvec[n]

Input: the coefficients of the objective function when the problem is of type \texttt{Nag\_LP}. The problem type is specified by the optional parameter \texttt{prob} (see Section 8) and the values \texttt{Nag\_LP} and \texttt{Nag\_FP} represent linear programming problem and feasible point problem respectively. \texttt{Nag\_LP} is the default problem type for \texttt{nag\_opt\_lp}.

If the problem type \texttt{Nag\_FP} is specified then \texttt{cvec} is not referenced and a NULL pointer may be given.

x[n]

Input: an initial estimate of the solution.
Output: the point at which \texttt{nag\_opt\_lp} terminated. If \texttt{fail.code} = \texttt{NE\_NOERROR}, \texttt{NW\_SOLN\_NOT\_UNIQUE} or \texttt{NW\_NOT\_FEASIBLE}, \( x \) contains an estimate of the solution.

objf

Output: the value of the objective function at \( x \) if \( x \) is feasible, or the sum of infeasibilities at \( x \) otherwise. If the problem is of type \texttt{Nag\_FP} and \( x \) is feasible, \texttt{objf} is set to zero.

options

Input/Output: a pointer to a structure of type \texttt{Nag\_E04\_Opt} whose members are optional parameters for \texttt{nag\_opt\_lp}. These structure members offer the means of adjusting some of the parameter values of the algorithm and on output will supply further details of the results.

A description of the members of \texttt{options} is given below in Section 8. Some of the results returned in \texttt{options} can be used by \texttt{nag\_opt\_lp} to perform a ‘warm start’ if it is re-entered (see the member \texttt{start} in Section 8.2).

If any of these optional parameters are required then the structure \texttt{options} should be declared and initialised by a call to \texttt{nag\_opt\_init (e04xxc)} and supplied as an argument to \texttt{nag\_opt\_lp}. However, if the optional parameters are not required the NAG defined null pointer, \texttt{E04\_DEFAULT}, can be used in the function call.

comm

Input/Output: structure containing pointers for user communication with an optional user defined printing function. See Section 8.3.1 for details. If the user does not need to make use of this communication feature then the null pointer \texttt{NAGCOMM\_NULL} may be used in the call to \texttt{nag\_opt\_lp}.

fail

The NAG error parameter, see the Essential Introduction to the NAG C Library.
Users are recommended to declare and initialise \texttt{fail} and set \texttt{fail.print} = \texttt{TRUE} for this function. \texttt{nag\_opt\_lp} returns with \texttt{fail.code} = \texttt{NE\_NOERROR} if \( x \) is a strong local minimizer, i.e., the reduced gradient is negligible and the Lagrange multipliers are optimal.

4.1. Description of Printed Output

Intermediate and final results are printed out by default. The level of printed output can be controlled by the user with the structure member \texttt{options.print\_level} (see Section 8.2). The default print level of \texttt{Nag\_Solv\_Iter} provides a single line of output at each iteration and the final result. This section describes the default printout produced by \texttt{nag\_opt\_lp}.

The convention for numbering the constraints in the iteration results is that indices 1 to \( n \) refer to the bounds on the variables, and indices \( n + 1 \) to \( n + m_{\text{lin}} \) refer to the general constraints.
When the status of a constraint changes, the index of the constraint is printed, along with the designation \( L \) (lower bound), \( U \) (upper bound), \( E \) (equality), \( F \) (temporarily fixed variable) or \( A \) (artificial constraint).
The single line of intermediate results output on completion of each iteration gives:

- **Itn** the iteration count.
- **Jdel** the index of the constraint deleted from the working set. If **Jdel** is zero, no constraint was deleted.
- **Jadd** the index of the constraint added to the working set. If **Jadd** is zero, no constraint was added.
- **Step** the step taken along the computed search direction. If a constraint is added during the current iteration (i.e., **Jadd** is positive), **Step** will be the step to the nearest constraint. When the problem is of type **Nag LP** the step can be greater than 1.0 during the optimality phase.
- **Ninf** the number of violated constraints (infeasibilities). This will be zero during the optimality phase.
- **Sinf/Obj** the value of the current objective function. If **x** is not feasible, **Sinf** gives a weighted sum of the magnitudes of constraint violations. If **x** is feasible, **Obj** is the value of the objective function. The output line for the final iteration of the feasibility phase (i.e., the first iteration for which **Ninf** is zero) will give the value of the true objective at the first feasible point.

During the optimality phase, the value of the objective function will be non-increasing. During the feasibility phase, the number of constraint infeasibilities will not increase until either a feasible point is found, or the optimality of the multipliers implies that no feasible point exists. Once optimal multipliers are obtained, the number of infeasibilities can increase, but the sum of infeasibilities will either remain constant or be reduced until the minimum sum of infeasibilities is found.

- **Bnd** the number of simple bound constraints in the current working set.
- **Lin** the number of general linear constraints in the current working set.
- **Nart** the number of artificial constraints in the working set.
- **Nrz** the dimension of the subspace in which the objective function is currently being minimized. The value of **Nrz** is the number of variables minus the number of constraints in the working set; i.e., **Nrz** = \( n - (\text{Bnd} + \text{Lin} + \text{Nart}) \).
- **Norm Gz** the Euclidean norm of the reduced gradient. During the optimality phase, this norm will be approximately zero after a unit step.

The printout of the final result consists of:

- **Varbl** the name \( V \) and index \( j \), for \( j = 1, 2, \ldots, n \) of the variable.
- **State** the state of the variable (FR if neither bound is in the working set, EQ if a fixed variable, LL if on its lower bound, UL if on its upper bound, TF if temporarily fixed at its current value). If **Value** lies outside the upper or lower bounds by more than the feasibility tolerance, **State** will be ++ or -- respectively.
- **Value** the value of the variable at the final iteration.
- **Lower bound** the lower bound specified for the variable. (None indicates that \( \text{bl}[j-1] \leq -\text{inf bound}. \)
- **Upper bound** the upper bound specified for the variable. (None indicates that \( \text{bu}[j-1] \geq \text{inf bound}. \)
nag_opt_lp

Lagr mult  the value of the Lagrange multiplier for the associated bound constraint. This will be zero if State is FR. If x is optimal, the multiplier should be non-negative if State is LL, and non-positive if State is UL.

Residual  the difference between the variable Value and the nearer of its bounds bl[j – 1] and bu[j – 1].

The meaning of the printout for general constraints is the same as that given above for variables, with ‘variable’ replaced by ‘constraint’, and with the following change in the heading:

LCon  the name (L) and index j, for j = 1, 2, …, mlin of the constraint.

5. Comments

A list of possible error exits and warnings from nag_opt_lp is given in Section 9. Scaling and accuracy are considered in Section 10.

6. Example 1

This example minimizes the function

\[-0.02x_1 - 0.2x_2 - 0.2x_3 - 0.2x_4 - 0.2x_5 + 0.04x_6 + 0.04x_7\]

subject to the bounds

\[-0.01 \leq x_1 \leq 0.01\]
\[-0.10 \leq x_2 \leq 0.15\]
\[-0.01 \leq x_3 \leq 0.03\]
\[-0.04 \leq x_4 \leq 0.02\]
\[-0.10 \leq x_5 \leq 0.05\]
\[-0.01 \leq x_6\]
\[-0.01 \leq x_7\]

and the general constraints

\[x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = -0.13\]
\[0.15x_1 + 0.04x_2 + 0.02x_3 + 0.04x_4 + 0.02x_5 + 0.01x_6 + 0.03x_7 \leq -0.0049\]
\[0.03x_1 + 0.05x_2 + 0.08x_3 + 0.02x_4 + 0.06x_5 + 0.01x_6 \leq -0.0064\]
\[0.02x_1 + 0.04x_2 + 0.01x_3 + 0.02x_4 + 0.02x_5 \leq -0.0037\]
\[0.02x_1 + 0.03x_2 + 0.01x_5 \leq -0.0012\]
\[-0.0992 \leq 0.70x_1 + 0.75x_2 + 0.80x_3 + 0.75x_4 + 0.80x_5 + 0.97x_6\]
\[-0.003 \leq 0.02x_1 + 0.06x_2 + 0.08x_3 + 0.12x_4 + 0.02x_5 + 0.01x_6 + 0.97x_7 \leq 0.002\]

The initial point, which is infeasible, is

\[x_0 = (-0.01, -0.03, 0.0, -0.01, -0.1, 0.02, 0.01)^T.\]

The computed solution (to five figures) is

\[x^* = (-0.01, -0.1, 0.03, 0.02, -0.067485, -0.0022801, -0.00023453)^T.\]

Four bound constraints and three general constraints are active at the solution.

This first example shows the simple use of nag_opt_lp where default values are used for all optional parameters. A second example showing the use of optional parameters is given in Section 13. There is one example program file, the main program of which calls both examples. The main program and example 1 are given below.
6.1. Program Text

/* nag_opt_lp (e04mfc) Example Program
 * Mark 6 revised, 2000.
 */

#include <nag.h>
#include <stdio.h>
#include <nag_stdlib.h>
#include <nag_string.h>
#include <nage04.h>

static void ex1(void);
static void ex2(void);

#define MAXN 10
#define MAXLIN 7
#define MAXBND MAXN+MAXLIN

main(void)
{
    /* Two examples are called, ex1() uses the
     * default settings to solve a problem while
     * ex2() solves another problem with some
     * of the optional parameters set by the user.
     */

    Vprintf("e04mfc Example Program Results.\n");
    ex1();
    ex2();
    exit(EXIT_SUCCESS);
}

static void ex1()
{
    double x[MAXN], cvec[MAXN];
    double a[MAXLIN][MAXN];
    double bl[MAXBND], bu[MAXBND];
    double objf;
    Integer i, j, n, nbnd, tda, nclin;
    static NagError fail;

    Vprintf("nExample 1: default options used.\n");
    Vscanf(" %*[^\n]"); /* Skip headings in data file */
    Vscanf(" %*[^-\n]"); /* Skip headings in data file */

    fail.print = TRUE;

    /* Set the actual problem dimensions.  
     * n = the number of variables.  
     * nclin = the number of general linear constraints (may be 0).  
     * nbnd = the number of variables + linear constraints  
     */
    tda = MAXN;
    n = 7;
    nclin = 7;
    nbnd = n + nclin;

    /* cvec = the objective function coefficients.  
     * a = the linear constraint matrix.  
     * bl = the lower bounds on x and A*x.  
     * bu = the upper bounds on x and A*x.  
     * x = the initial estimate of the solution.  
     */
}
/* Read the objective function coefficients */
Vscanf(" %*[\n]"); /* Skip heading in data file */
for (i = 0; i < n; ++i)
    Vscanf("%lf", &cvec[i]);

/* Read the linear constraint matrix A. */
Vscanf(" %*[\n]"); /* Skip heading in data file */
for (i = 0; i < nclin; ++i)
    for (j = 0; j < n; ++j)
        Vscanf("%lf", &a[i][j]);

/* Read the bounds. */
for (i = 0; i < n + nclin; ++i)
    if (i < n)
        Vscanf("%lf", &bl[i]);
    else
        Vscanf("%lf", &bu[i]);

/* Read the initial estimate of x. */
for (i = 0; i < n; ++i)
    Vscanf("%lf", &x[i]);

/* Solve the problem. */
e04mfc(n, nclin, (double *)a, tda, bl, bu, cvec, x, &objf, E04_DEFAULT, NAGCOMM_NULL, &fail);
if (fail.code != NE_NOERROR) exit(EXIT_FAILURE);
}

6.2. Program Data

e04mfc Example Program Data

Data for example 1.

Objective function coefficients
-0.02 -0.2 -0.2 -0.2 -0.2 0.04 0.04

Linear constraint matrix, A.
1.0 1.0 1.0 1.0 1.0 1.0
0.15 0.04 0.04 0.04 0.02 0.01 0.03
0.03 0.05 0.08 0.02 0.06 0.01 0.05
0.02 0.04 0.01 0.02 0.02 0.0 0.0
0.02 0.03 0.0 0.0 0.01 0.0 0.0
0.70 0.75 0.80 0.75 0.80 0.97 0.0
0.02 0.06 0.08 0.12 0.02 0.01 0.01

Lower bounds
-0.01 -0.1 -0.01 -0.04 -0.1 -0.01
-0.13 -1.0e21 -1.0e21 -1.0e21 -1.0e21 -0.0992 -0.003

Upper bounds
0.01 0.15 0.03 0.02 0.05 1.0e21 1.0e21
-0.13 -0.0049 -0.0064 -0.0037 -0.0012 1.0e21 0.002

Initial estimate of x
-0.01 -0.03 0.0 -0.01 -0.1 0.02 0.01
6.3. Program Results

**e04mfc Example Program Results.**

Example 1: default options used.

**Parameters to e04mfc**

---

**Linear constraints**

<table>
<thead>
<tr>
<th>prob</th>
<th>start</th>
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</thead>
<tbody>
<tr>
<td>Nag_LP</td>
<td>Nag_Cold</td>
</tr>
</tbody>
</table>

**Number of variables**

7

---

**ftol**

1.05e-08

**fcheck**

50

**inf_bound**

1.00e+20

**max_iter**

70

**optim_tol**

1.72e-13

**print_level**

Nag_Soln_Iter

**outfile**

stdout

---

**Memory allocation:**

state

Nag

ax

Nag

lambda

Nag

---

**Results from e04mfc:**

---

<table>
<thead>
<tr>
<th>Itn</th>
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<th>Jadd</th>
<th>Step</th>
<th>Ninf</th>
<th>Sinf/Obj</th>
<th>Bnd</th>
<th>Lin</th>
<th>Nart</th>
<th>Nrz</th>
<th>Norm Gz</th>
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**Final solution:**

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<th>Varbl</th>
<th>State</th>
<th>Value</th>
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<th>Upper Bound</th>
<th>Lagr Mult</th>
<th>Residual</th>
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<table>
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</table>

Exit after 7 iterations.

Optimal LP solution found.

**Final LP objective value = 2.3596482e-02**
7. Further Description

This section gives a detailed description of the algorithm used in nag_opt_lp. This, and possibly the next section, Section 8, may be omitted if the more sophisticated features of the algorithm and software are not currently of interest.

7.1. Overview

nag_opt_lp is based on an inertia-controlling method due to Gill and Murray (1978) and is described in detail by Gill et al (1991a). Here the main features of the method are summarized. Where possible, explicit reference is made to the names of variables that are parameters of nag_opt_lp or appear in the printed output. nag_opt_lp has two phases: finding an initial feasible point by minimizing the sum of infeasibilities (the feasibility phase), and minimizing the linear objective function within the feasible region (the optimality phase). The computations in both phases are performed by the same functions. The two-phase nature of the algorithm is reflected by changing the function being minimized from the sum of infeasibilities to the linear objective function. The feasibility phase does not perform the standard simplex method (i.e., it does not necessarily find a vertex), except in the LP case when \( m_{\text{lin}} \leq n \). Once any iterate is feasible, all subsequent iterates remain feasible.

In general, an iterative process is required to solve a linear program. (For simplicity, we shall always consider a typical iteration and avoid reference to the index of the iteration.) Each new iterate \( \bar{x} \) is defined by

\[
\bar{x} = x + \alpha p,
\]

where the steplength \( \alpha \) is a non-negative scalar, and \( p \) is called the search direction.

At each point \( x \), a working set of constraints is defined to be a linearly independent subset of the constraints that are satisfied 'exactly' (to within the tolerance defined by the optional parameter fto; see Section 8.2). The working set is the current prediction of the constraints that hold with equality at a solution of an LP problem. The search direction is constructed so that the constraints in the working set remain unaltered for any value of the step length. For a bound constraint in the working set, this property is achieved by setting the corresponding component of the search direction to zero. Thus, the associated variable is fixed and the specification of the working set induces a partition of \( x \) into fixed and free variables. During a given iteration, the fixed variables are effectively removed from the problem; since the relevant components of the search direction are zero, the columns of \( A \) corresponding to fixed variables may be ignored.

Let \( m_w \) denote the number of general constraints in the working set and let \( n_{fz} \) denote the number of variables fixed at one of their bounds (\( m_w \) and \( n_{fz} \) are the quantities Lin and Bnd in the printed output from nag_opt_lp). Similarly, let \( n_{fr} (n_{fr} = n - n_{fz}) \) denote the number of free variables. At every iteration, the variables are re-ordered so that the last \( n_{fz} \) variables are fixed, with all other relevant vectors and matrices ordered accordingly.

7.2. Definition of the Search Direction

Let \( A_{fr} \) denote the \( m_w \) by \( n_{fr} \) sub-matrix of general constraints in the working set corresponding to the free variables, and let \( p_{fr} \) denote the search direction with respect to the free variables only. The general constraints in the working set will be unaltered by any move along \( p \) if

\[
A_{fr}p_{fr} = 0.
\]

In order to compute \( p_{fr} \), the \( TQ \) factorization of \( A_{fr} \) is used:

\[
A_{fr}Q_{fr} = (0 \ T),
\]

where \( T \) is a non-singular \( m_w \) by \( m_w \) upper triangular matrix (i.e., \( t_{ij} = 0 \) if \( i > j \)), and the non-singular \( n_{fr} \) by \( n_{fr} \) matrix \( Q_{fr} \) is the product of orthogonal transformations (see Gill et al (1984)). If the columns of \( Q_{fr} \) are partitioned so that

\[
Q_{fr} = (Z \ Y),
\]

then

\[
A_{fr}p_{fr} = Z(0 \ Y),
\]

and

\[
(0 \ Y)p_{fr} = 0,
\]

where \( Y \) is the \( n_{fr} \) by \( n_{fr} \) upper triangular matrix

\[
(0 \ Y) = (0 \ I) + Z Q_{fr}.
\]
where \( Y \) is \( n \times m_w \), then the \( n_z \) \( (n_z = n - m_w) \) columns of \( Z \) form a basis for the null space of \( A_{r} \). Let \( n_z \) be an integer such that \( 0 \leq n_z \leq n \), and let \( Z_r \) denote a matrix whose \( n_z \) columns are a subset of the columns of \( Z \). (The integer \( n_z \) is the quantity \( \text{Nrz} \) in the printed output from nag_opt_lp. In many cases, \( Z_r \) will include all the columns of \( Z \).) The direction \( p_{fr} \) will satisfy (2) if

\[
p_{fr} = Z_r p_r,
\]

where \( p_r \) is any \( n_r \)-vector.

### 7.3. The Main Iteration

Let \( Q \) denote the \( n \) by \( n \) matrix

\[
Q = \begin{pmatrix} Q_{fr} & I_{fx} \end{pmatrix},
\]

where \( I_{fx} \) is the identity matrix of order \( n_{fx} \). Let \( g_q \) denote the transformed gradient

\[
g_q = QTc
\]

and let the vector of first \( n_c \) elements of \( g_q \) be denoted by \( g_c \). The quantity \( g_c \) is known as the reduced gradient of \( c^T x \). If the reduced gradient is zero, \( x \) is a constrained stationary point in the subspace defined by \( Z \). During the feasibility phase, the reduced gradient will usually be zero only at a vertex (although it may be zero at non-vertices in the presence of constraint dependencies). During the optimality phase, a zero reduced gradient implies that \( x \) minimizes the linear objective when the constraints in the working set are treated as equalities. At a constrained stationary point, Lagrange multipliers \( \lambda_c \) and \( \lambda_b \) for the general and bound constraints are defined from the equations

\[
A_{fr}^T \lambda_c = g_{fr} \quad \text{and} \quad \lambda_b = g_{fx} - A_{fr}^T \lambda_c.
\]

Given a positive constant \( \delta \) of the order of the machine precision, a Lagrange multiplier \( \lambda_j \) corresponding to an inequality constraint in the working set is said to be optimal if \( \lambda_j \leq \delta \) when the associated constraint is at its upper bound, or if \( \lambda_j \geq -\delta \) when the associated constraint is at its lower bound. If a multiplier is non-optimal, the objective function (either the true objective or the sum of infeasibilities) can be reduced by deleting the corresponding constraint (with index \( J_{de1} \); see Section 8.3) from the working set.

If optimal multipliers occur during the feasibility phase and the sum of infeasibilities is non-zero, there is no feasible point, and nag_opt_lp will continue until the minimum value of the sum of infeasibilities has been found. At this point, the Lagrange multiplier \( \lambda_j \) corresponding to an inequality constraint in the working set will be such that \((1 + \delta) \leq \lambda_j \leq \delta \) when the associated constraint is at its upper bound, and \( -\delta \leq \lambda_j \leq (1 + \delta) \) when the associated constraint is at its lower bound. Lagrange multipliers for equality constraints will satisfy \(|\lambda_j| \leq 1 + \delta \).

If the reduced gradient is not zero, Lagrange multipliers need not be computed and the non-zero elements of the search direction \( p \) are given by \( Z_r p_r \). The choice of step length is influenced by the need to maintain feasibility with respect to the satisfied constraints.

Each change in the working set leads to a simple change to \( A_{fr} \): if the status of a general constraint changes, a row of \( A_{fr} \) is altered; if a bound constraint enters or leaves the working set, a column of \( A_{fr} \) changes. Explicit representations are recorded of the matrices \( T \) and \( Q_{fr} \), and of vectors \( QTg \), and \( QTc \).

One of the most important features of nag_opt_lp is its control of the conditioning of the working set, whose nearness to linear dependence is estimated by the ratio of the largest to smallest diagonal elements of the \( TQ \) factor \( T \) (the printed value \text{Cond} \( T \); see Section 8.3). In constructing the initial working set, constraints are excluded that would result in a large value of \text{Cond} \( T \).

nag_opt_lp includes a rigorous procedure that prevents the possibility of cycling at a point where the active constraints are nearly linearly dependent (see Gill et al (1989)). The main feature of the anti-cycling procedure is that the feasibility tolerance is increased slightly at the start of every iteration. This not only allows a positive step to be taken at every iteration, but also provides, whenever
possible, a choice of constraints to be added to the working set. Let $\alpha_m$ denote the maximum step at which $x + \alpha_m p$ does not violate any constraint by more than its feasibility tolerance. All constraints at a distance $\alpha$ ($\alpha \leq \alpha_m$) along $p$ from the current point are then viewed as acceptable candidates for inclusion in the working set. The constraint whose normal makes the largest angle with the search direction is added to the working set.

7.4. Choosing the Initial Working Set

Let $Z$ be partitioned as $Z = (Z_r, Z_a)$. A working set for which $Z_r$ defines the null space can be obtained by including the rows of $Z_a^T$ as ‘artificial constraints’. Minimization of the objective function then proceeds within the subspace defined by $Z_r$, as described in Section 7.2.

The artificially augmented working set is given by

$$\bar{A}_{fr} = \begin{pmatrix} Z_a^T \\ A_{fr} \end{pmatrix},$$
(6)

so that $p_{fr}$ will satisfy $A_{fr} p_{fr} = 0$ and $Z_a^T p_{fr} = 0$. By definition of the $TQ$ factorization, $\bar{A}_{fr}$ automatically satisfies the following:

$$\bar{A}_{fr} Q_{fr} = \begin{pmatrix} Z_a^T \\ A_{fr} \end{pmatrix} Q_{fr} = \begin{pmatrix} Z_a^T \\ A_{fr} \end{pmatrix} (Z_r, Z_a, Y) = (0, \bar{T}),$$

where

$$\bar{T} = \begin{pmatrix} I & 0 \\ 0 & T \end{pmatrix},$$

and hence the $TQ$ factorization of (6) is available trivially from $T$ and $Q_{fr}$ without additional expense.

The matrix $Z_a$ is not kept fixed, since its role is purely to define an appropriate null space; the $TQ$ factorization can therefore be updated in the normal fashion as the iterations proceed. No work is required to ‘delete’ the artificial constraints associated with $Z_a$ when $Z_r^T g_{fr} = 0$, since this simply involves repartitioning $Q_{fr}$. The ‘artificial’ multiplier vector associated with the rows of $Z_a$ is equal to $Z_a^T g_{fr}$, and the multipliers corresponding to the rows of the ‘true’ working set are the multipliers that would be obtained if the artificial constraints were not present. If an artificial constraint is ‘deleted’ from the working set, an $A$ appears alongside the entry in the Jdel column of the printed output (see Section 8.3).

The number of columns in $Z_a$ and $Z_r$ and the Euclidean norm of $Z_r^T g_{fr}$, appear in the printed output as $\text{Nart}$, $\text{Nrz}$ and $\text{Norm Gz}$ (see Section 8.3).

Under some circumstances, a different type of artificial constraint is used when solving a linear program. Although the algorithm of nag_opt_lp does not usually perform simplex steps (in the traditional sense), there is one exception: a linear program with fewer general constraints than variables (i.e., $m_{lin} \leq n$). (Use of the simplex method in this situation leads to savings in storage.) At the starting point, the ‘natural’ working set (the set of constraints exactly or nearly satisfied at the starting point) is augmented with a suitable number of ‘temporary’ bounds, each of which has the effect of temporarily fixing a variable at its current value. In subsequent iterations, a temporary bound is treated as a standard constraint until it is deleted from the working set, in which case it is never added again. If a temporary bound is ‘deleted’ from the working set, an $F$ (for ‘Fixed’) appears alongside the entry in the Jdel column of the printed output (see Section 8.3).

8. Optional Parameters

A number of optional input and output parameters to nag_opt_lp are available through the structure argument options, type Nag_E04_Opt. A parameter may be selected by assigning an appropriate value to the relevant structure member; those parameters not selected will be assigned default values. If no use is to be made of any of the optional parameters the user should use the NAG defined null pointer, E04_DEFAULT, in place of options when calling nag_opt_lp; the default settings will then be used for all parameters.
Before assigning values to options directly the structure must be initialised by a call to the function nag_opt_init (e04xzc). Values may then be assigned to the structure members in the normal C manner.

Option settings may also be read from a file using the function nag_opt_read (e04xyc) in which case initialisation of the options structure will be performed automatically if not already done. Any subsequent direct assignment to the options structure must not be preceded by initialisation.

If assignment of functions and memory to pointers in the options structure is required, this must be done directly in the calling program; they cannot be assigned using nag_opt_read (e04xyc).

8.1. Optional Parameter Checklist and Default Values

For easy reference, the following list shows the members of options which are valid for nag_opt_lp together with their default values where relevant. The number \( \epsilon \) is a generic notation for machine precision (see nag_machine_precision (X02AJC)).

<table>
<thead>
<tr>
<th>Member</th>
<th>Default Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nag_ProblemType prob</td>
<td>Nag_LP</td>
</tr>
<tr>
<td>Nag_Start start</td>
<td>Nag_Cold</td>
</tr>
<tr>
<td>Boolean list</td>
<td>TRUE</td>
</tr>
<tr>
<td>Nag_PrintType print_level</td>
<td>Nag_Soln_Iter</td>
</tr>
<tr>
<td>char outfile[80]</td>
<td>stdout</td>
</tr>
<tr>
<td>void (*print_fun)()</td>
<td>NULL</td>
</tr>
<tr>
<td>Integer max_iter</td>
<td>max(50, 5(n+nclin))</td>
</tr>
<tr>
<td>double crash_tol</td>
<td>0.01</td>
</tr>
<tr>
<td>double ftol</td>
<td>( \sqrt{\epsilon} )</td>
</tr>
<tr>
<td>double optim_tol</td>
<td>( \epsilon^{0.8} )</td>
</tr>
<tr>
<td>Integer reset_ftol</td>
<td>10000</td>
</tr>
<tr>
<td>Integer fcheck</td>
<td>50</td>
</tr>
<tr>
<td>double inf_bound</td>
<td>10^{20}</td>
</tr>
<tr>
<td>double inf_step</td>
<td>\max(inf_bound, 10^{20})</td>
</tr>
<tr>
<td>Integer *state</td>
<td>size n+nclin</td>
</tr>
<tr>
<td>double *ax</td>
<td>size nclin</td>
</tr>
<tr>
<td>double *lambda</td>
<td>size n+nclin</td>
</tr>
<tr>
<td>Integer iter</td>
<td></td>
</tr>
</tbody>
</table>

8.2. Description of Optional Parameters

**prob** – Nag_ProblemType

Input: specifies the problem type. The following are the two possible values of prob and the size of the array cvec that is required to define the objective function:

- **Nag_FP** cvec not accessed;
- **Nag_LP** cvec[n] required;

Nag_FP denotes a feasible point problem and Nag_LP a linear programming problem. Constraint: **options.prob = Nag_FP** or **Nag_LP**.

**start** – Nag_Start

Input: specifies how the initial working set is chosen. With **options.start = Nag_Cold**, nag_opt_lp chooses the initial working set based on the values of the variables and constraints at the initial point. Broadly speaking, the initial working set will include equality constraints and bounds or inequality constraints that violate or ‘nearly’ satisfy their bounds (to within crash_tol; see below).

With **options.start = Nag_Warm**, the user must provide a valid definition of every element of the array pointer options.state (see below for the definition of this member of options). nag_opt_lp will override the users’ specification of state if necessary, so that a poor choice of the working set will not cause a fatal error. Nag_Warm will be advantageous if a good estimate of the initial working set is available – for example, when nag_opt_lp is called repeatedly to solve related problems. Constraint: **options.start = Nag_Cold** or **Nag_Warm**.
list – Boolean
Default = TRUE
Input: if options.list = TRUE the parameter settings in the call to nag_opt_lp will be printed.

print_level – Nag_PrintType
Default = Nag_Soln_Iter
Input: the level of results printout produced by nag_opt_lp. The following values are available.

| Nag_NoPrint  | No output.                          |
| Nag_Soln     | The final solution.                 |
| Nag_Iter     | One line of output for each iteration. |
| Nag_Iter_Long| A longer line of output for each iteration (line exceeds 80 characters). |
| Nag_Soln_Iter| The final solution and one line of output for each iteration (line exceeds 80 characters). |
| Nag_Soln_Iter_Const | As Nag_Soln_Iter_Long with the Lagrange multipliers, the variables x, the constraint values Ax and the constraint status also printed at each iteration. |
| Nag_Soln_Iter_Full | As Nag_Soln_Iter_Const with the diagonal elements of the upper triangular matrix T associated with the TQ factorization (3) of the working set. |

Details of each level of results printout are described in Section 8.3.
Constraint: options.print_level = Nag_NoPrint or Nag_Soln or Nag_Iter or Nag_Soln_Iter or Nag_Iter_Long or Nag_Soln_Iter_Long or Nag_Soln_Iter_Const or Nag_Soln_Iter_Full.

outfile – char[80]
Default = stdout
Input: the name of the file to which results should be printed. If options.outfile[0] = ‘\0’ then the stdout stream is used.

print_fun – pointer to function
Default = NULL
Input: printing function defined by the user; the prototype of print_fun is

```c
void (*print_fun)(const Nag_Search_State *st, Nag_Comm *comm);
```

See Section 8.3.1 below for further details.

max_iter – Integer
Default = max (50, 5(n + nclin))
Input: max_iter specifies the maximum number of iterations to be performed by nag_opt_lp.

If the user wishes to check that a call to nag_opt_lp is correct before attempting to solve the problem in full then max_iter may be set to 0. No iterations will then be performed but the initialisation stages prior to the first iteration will be processed and a listing of parameter settings output if options.list = TRUE (the default setting).
Constraint: options.max_iter ≥ 0.

crash_tol – double
Default = 0.01
Input: crash_tol is used in conjunction with the optional parameter start. When options.start has the default setting, i.e., options.start = Nag_Cold, nag_opt_lp selects an initial working set. The initial working set will include bounds or general inequality constraints that lie within crash_tol of their bounds. In particular, a constraint of the form $a_j^T x \geq l$ will be included in the initial working set if $|a_j^T x - l| \leq \text{crash_tol} \times (1 + |l|)$.
Constraint: $0.0 \leq \text{options.crash_tol} \leq 1.0$.

ftol – double
Default = $\sqrt{\epsilon}$
Input: ftol defines the maximum acceptable absolute violation in each constraint at a ‘feasible’ point. For example, if the variables and the coefficients in the general constraints are of order unity, and the latter are correct to about 6 decimal digits, it would be appropriate to specify ftol as $10^{-6}$.
	nag_opt_lp attempts to find a feasible solution before optimizing the objective function. If the sum of infeasibilities cannot be reduced to zero, nag_opt_lp finds the minimum value of
the sum. Let $S_{\text{inf}}$ be the corresponding sum of infeasibilities. If $S_{\text{inf}}$ is quite small, it may be appropriate to raise $\texttt{ftol}$ by a factor of 10 or 100. Otherwise, some error in the data should be suspected.

Note that a ‘feasible solution’ is a solution that satisfies the current constraints to within the tolerance $\texttt{ftol}$.

Constraint: $\texttt{options.ftol} > 0.0$.

$\texttt{optim\_tol}$ – double

Input: $\texttt{options.optim\_tol}$ defines the tolerance used to determine whether the bounds and generated constraints have the correct sign for the solution to be judged optimal.

$\texttt{reset\_ftol}$ – Integer

Input: this option is part of an anti-cycling procedure designed to guarantee progress even on highly degenerate problems.

The strategy is to force a positive step at every iteration, at the expense of violating the constraints by a small amount. Suppose that the value of the optional parameter $\texttt{ftol}$ is $\delta$. Over a period of $\texttt{reset\_ftol}$ iterations, the feasibility tolerance actually used by nag\_opt\_lp increases from $0.5\delta$ to $\delta$ (in steps of $0.5\delta/\texttt{reset\_ftol}$).

At certain stages the following ‘resetting procedure’ is used to remove constraint infeasibilities. First, all variables whose upper or lower bounds are in the working set are moved exactly onto their bounds. A count is kept of the number of nontrivial adjustments made. If the count is positive, iterative refinement is used to give variables that satisfy the working set to (essentially) machine precision. Finally, the current feasibility tolerance is reinitialized to $0.5\delta$.

If a problem requires more than $\texttt{reset\_ftol}$ iterations, the resetting procedure is invoked and a new cycle of $\texttt{reset\_ftol}$ iterations is started. (The decision to resume the feasibility phase or optimality phase is based on comparing any constraint infeasibilities with $\delta$.)

The resetting procedure is also invoked when nag\_opt\_lp reaches an apparently optimal, infeasible or unbounded solution, unless this situation has already occurred twice. If any nontrivial adjustments are made, iterations are continued.

Constraint: $0 < \texttt{options.reset\_ftol} < 10000000$.

$\texttt{fcheck}$ – Integer

Input: every $\texttt{fcheck}$ iterations, a numerical test is made to see if the current solution $x$ satisfies the constraints in the working set. If the largest residual of the constraints in the working set is judged to be too large, the current working set is re-factorized and the variables are recomputed to satisfy the constraints more accurately.

Constraint: $\texttt{options.fcheck} \geq 1$.

$\texttt{inf\_bound}$ – double

Input: $\texttt{inf\_bound}$ defines the ‘infinite’ bound in the definition of the problem constraints. Any upper bound greater than or equal to $\texttt{inf\_bound}$ will be regarded as plus infinity (and similarly for a lower bound less than or equal to $-\texttt{inf\_bound}$).

Constraint: $\texttt{options.inf\_bound} > 0.0$.

$\texttt{inf\_step}$ – double

Input: $\texttt{inf\_step}$ specifies the magnitude of the change in variables that will be considered a step to an unbounded solution. (Note that an unbounded solution can occur only when the problem is of type Nag\_LP). If the change in $x$ during an iteration would exceed the value of $\texttt{inf\_step}$, the objective function is considered to be unbounded below in the feasible region.

Constraint: $\texttt{options.inf\_step} > 0.0$. 


Input: state need not be set if the default option of options.start = Nag_Cold is used as 
n+nclin values of memory will be automatically allocated by nag_opt_lp.

If the option start = Nag_Warm has been chosen, state must point to a minimum of n+nclin
elements of memory. This memory will already be available if the options structure has been
used in a previous call to nag_opt_lp from the calling program, using the same values of n and
nclin and start = Nag_Cold. If a previous call has not been made sufficient memory must be
allocated to state by the user.

When a warm start is chosen state should specify the desired status of the constraints at the
start of the feasibility phase. More precisely, the first n elements of state refer to the upper
and lower bounds on the variables, and the next m_lin elements refer to the general linear
constraints (if any). Possible values for state[j] are as follows:

<table>
<thead>
<tr>
<th>state[j]</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>The corresponding constraint should not be in the initial working set.</td>
</tr>
<tr>
<td>1</td>
<td>The constraint should be in the initial working set at its lower bound.</td>
</tr>
<tr>
<td>2</td>
<td>The constraint should be in the initial working set at its upper bound.</td>
</tr>
<tr>
<td>3</td>
<td>The constraint should be in the initial working set as an equality. This value should only be specified if bl[j] = bu[j]. The values 1, 2 or 3 all have the same effect when bl[j] = bu[j].</td>
</tr>
</tbody>
</table>

The values -2, -1 and 4 are also acceptable but will be reset to zero by the function. In
particular, if nag_opt_lp has been called previously with the same values of n and nclin, state
already contains satisfactory information. (See also the description of the optional parameter
start). The function also adjusts (if necessary) the values supplied in x to be consistent with
the values supplied in state.

Output: If nclin > 0, ax points to the final values of the linear constraints Ax.
**e04 – Minimizing or Maximizing a Function**

**lambda** – double *

Input: n+nclin values of memory will be automatically allocated by nag_opt_lp and this is the recommended method of use of options.lambda. However a user may supply memory from the calling program.

Output: the values of the Lagrange multipliers for each constraint with respect to the current working set. The first n elements contain the multipliers for the bound constraints on the variables, and the next mlin elements contain the multipliers for the general linear constraints (if any). If state[j] = 0 (i.e., constraint j is not in the working set), lambda[j] is zero. If x is optimal, lambda[j] should be non-negative if state[j] = 1, non-positive if state[j] = 2 and zero if state[j] = 4.

**iter** – Integer

Output: the total number of iterations performed in the feasibility phase and (if appropriate) the optimality phase.

### 8.3. Description of Printed Output

The level of printed output can be controlled by the user with the structure members options.list and options.print_level (see Section 8.2). If list = TRUE then the parameter values to nag_opt_lp are listed, whereas the printout of results is governed by the value of print_level. The default of print_level = Nag_Soln_Iters provides a single line of output at each iteration and the final result. This section describes all of the possible levels of results printout available from nag_opt_lp.

The convention for numbering the constraints in the iteration results is that indices 1 to n refer to the bounds on the variables, and indices n + 1 to n + mlin refer to the general constraints. When the status of a constraint changes, the index of the constraint is printed, along with the designation L (lower bound), U (upper bound), E (equality), F (temporarily fixed variable) or A (artificial constraint).

When print_level = Nag_Iters or Nag_Soln_Iters the following line of output is produced on completion of each iteration.

- **Itn** the iteration count.
- **Jdel** the index of the constraint deleted from the working set. If Jdel is zero, no constraint was deleted.
- **Jadd** the index of the constraint added to the working set. If Jadd is zero, no constraint was added.
- **Step** the step taken along the computed search direction. If a constraint is added during the current iteration (i.e., Jadd is positive), Step will be the step to the nearest constraint. During the optimality phase, the step can be greater than one only if the reduced Hessian is not positive-definite.
- **Ninf** the number of violated constraints (infeasibilities). This will be zero during the optimality phase.
- **Sinf/Obj** the value of the current objective function. If x is not feasible, Sinf gives a weighted sum of the magnitudes of constraint violations. If x is feasible, Obj is the value of the objective function. The output line for the final iteration of the feasibility phase (i.e., the first iteration for which Ninf is zero) will give the value of the true objective at the first feasible point.

During the optimality phase, the value of the objective function will be non-increasing. During the feasibility phase, the number of constraint infeasibilities will not increase until either a feasible point is found, or the optimality of the multipliers implies that no feasible point exists. Once optimal multipliers are obtained, the number of infeasibilities can increase, but the sum of infeasibilities will either remain constant or be reduced until the minimum sum of infeasibilities is found.

- **End** the number of simple bound constraints in the current working set.
Lin  the number of general linear constraints in the current working set.

Nart  the number of artificial constraints in the working set, i.e., the number of columns of $Z_a$ (see Section 7). At the start of the optimality phase, Nart provides an estimate of the number of nonpositive eigenvalues in the reduced Hessian.

Nrz  is the number of columns of $Z_r$ (see Section 7). Nrz is the number of variables minus the number of constraints in the working set; i.e., $Nrz = n - (Bnd + Lin + Nart)$.

The value of $n_z$, the number of columns of $Z$ (see Section 7) can be calculated as $n_z = n - (Bnd + Lin)$. A zero value of $n_z$ implies that $x$ lies at a vertex of the feasible region.

Norm Gz  $\|Z_r^T g_f\|$, the Euclidean norm of the reduced gradient with respect to $Z_r$. During the optimality phase, this norm will be approximately zero after a unit step.

If $\text{print}_level = \text{Nag}_\text{Iter}_\text{Long}, \text{Nag}_\text{Soln}_\text{Iter}_\text{Long}, \text{Nag}_\text{Soln}_\text{Iter}_\text{Const}$ or $\text{Nag}_\text{Soln}_\text{Iter}_\text{Full}$ the line of printout is extended to give the following information. (Note this longer line extends over more than 80 characters).

$\text{NOpt}$  is the number of non-optimal Lagrange multipliers at the current point. NOpt is not printed if the current $x$ is infeasible or no multipliers have been calculated. At a minimizer, NOpt will be zero.

Min LM  is the value of the Lagrange multiplier associated with the deleted constraint. If Min LM is negative, a lower bound constraint has been deleted; if Min LM is positive, an upper bound constraint has been deleted. If no multipliers are calculated during a given iteration, Min LM will be zero.

Cond T  is a lower bound on the condition number of the working set.

When $\text{options}_\text{print}_level = \text{Nag}_\text{Soln}_\text{Iter}_\text{Const}$ or $\text{Nag}_\text{Soln}_\text{Iter}_\text{Full}$ more detailed results are given at each iteration. For the setting $\text{Nag}_\text{Soln}_\text{Iter}_\text{Const}$ additional values output are:

Value of x  the value of $x$ currently held in $x$.

State  the current value of $\text{options}_\text{state}$ associated with $x$.

Value of Ax  the value of $Ax$ currently held in $\text{options}_\text{ax}$.

State  the current value of $\text{options}_\text{state}$ associated with $Ax$.

Also printed are the Lagrange Multipliers for the bound constraints, linear constraints and artificial constraints.

If $\text{print}_level = \text{Nag}_\text{Soln}_\text{Iter}_\text{Full}$ then the diagonal of $T$ and $Z_r$ are also output at each iteration.

When $\text{print}_level = \text{Nag}_\text{Soln}$, $\text{Nag}_\text{Soln}_\text{Iter}$, $\text{Nag}_\text{Soln}_\text{Iter}_\text{Const}$ or $\text{Nag}_\text{Soln}_\text{Iter}_\text{Full}$ the final printout from nag_opt_lp includes a listing of the status of every variable and constraint. The following describes the printout for each variable.

Varbl  the name ($V$) and index $j$, for $j = 1, 2, \ldots, n$ of the variable.

State  the state of the variable (FR if neither bound is in the working set, EQ if a fixed variable, LL if on its lower bound, UL if on its upper bound, TF if temporarily fixed at its current value). If Value lies outside the upper or lower bounds by more than the feasibility tolerance, State will be ++ or -- respectively.

Value  the value of the variable at the final iteration.

Lower bound  the lower bound specified for the variable.

(None indicates that $bl[j-1] \leq -\text{inf}_\text{bound}$.)
Upper bound  
the upper bound specified for the variable.
(None indicates that $bu[j - 1] \geq \text{inf_bound}$.)

Lagr mult  
the value of the Lagrange multiplier for the associated bound constraint. This will be zero if State is FR. If $x$ is optimal, the multiplier should be non-negative if State is LL, and non-positive if State is UL.

Residual  
the difference between the variable Value and the nearer of its bounds $bl[j - 1]$ and $bu[j - 1]$.

The meaning of the printout for general constraints is the same as that given above for variables, with ‘variable’ replaced by ‘constraint’, and with the following change in the heading:

LCon  
the name (L) and index $j$, for $j = 1, 2, \ldots, m_{\text{lin}}$ of the constraint.

### 8.3.1. Output of results via a user defined printing function

Users may also specify their own print function for output of iteration results and the final solution by use of the `options.print_fun` function pointer, which has prototype

```c
void (*print_fun)(const Nag_Search_State *st, Nag_Comm *comm);
```

The rest of this section can be skipped by a user who only wishes to use the default printing facilities.

When a user defined function is assigned to `options.print_fun` this will be called in preference to the internal print function of nag_opt_lp. Calls to the user defined function are again controlled by means of the `options.print_level` member. Information is provided through st and comm, the two structure arguments to `print_fun`.

If `comm->it_prt = TRUE` then the results from the last iteration of nag_opt_lp are set in the following members of st:

- **first** – Boolean
  - TRUE on the first call to `print_fun`.
- **iter** – Integer
  - the number of iterations performed.
- **n** – Integer
  - the number of variables.
- **nclin** – Integer
  - the number of linear constraints.
- **jdel** – Integer
  - index of constraint deleted.
- **jadd** – Integer
  - index of constraint added.
- **step** – double
  - the step taken along the current search direction.
- **ninf** – Integer
  - the number of infeasibilities.
- **f** – double
  - the value of the current objective function.
- **bnd** – Integer
  - number of bound constraints in the working set.
- **lin** – Integer
  - number of general linear constraints in the working set.
- **nart** – Integer
  - number of artificial constraints in the working set.
nrz – Integer
    number of columns of $Z_r$.

norm_gz – double
    Euclidean norm of the reduced gradient, $\|Z_r^T g_f\|$.

nopt – Integer
    number of non-optimal Lagrange multipliers.

min_lm – double
    value of the Lagrange multiplier associated with the deleted constraint.

condt – double
    a lower bound on the condition number of the working set.

x – double *
    x points to the n memory locations holding the current point $x$.

ax – double *
    ax points to the nclin memory locations holding the current values $Ax$.

state – Integer *
    state points to the n+nclin memory locations holding the status of the variables and general linear constraints. See Section 8.2 for a description of the possible status values.

t – double *
    the upper triangular matrix $T$ with st->lin columns. Matrix element $i,j$ is held in st->t[(i-1)*st->tdt + j - 1].

tdt – Integer
    the trailing dimension for st->t.

If comm->new_lm = TRUE then the Lagrange multipliers have been updated and the following members, kx, kactive, lambda and gq, are set:

kx – Integer *
    Indices of the bound constraints with associated multipliers. Value of st->kx[i] is the index of the constraint with multiplier st->lambda[i] for $i = 0,1,\ldots,\text{st->bnd}-1$.

kactive – Integer *
    Indices of the linear constraints with associated multipliers. Value of st->kactive[i] is the index of the constraint with multiplier st->lambda[st->bnd + i] for $i = 0,1,\ldots,\text{st->lin}-1$.

lambda – double *
    the multipliers for the constraints in the working set. lambda[i] for $i = 0,1,\ldots,\text{st->bnd}-1$ hold the multipliers for the bound constraints while the multipliers for the linear constraints are held at indices $i = \text{st->bnd},\ldots,\text{st->bnd} + \text{st->lin}-1$.

gq – double *
    st->gq[i] for $i = 0,1,\ldots,\text{st->nart} - 1$ hold the multipliers for the artificial constraints.

The following members of st are also relevant and apply when comm->it_prt or comm->new_lm is TRUE.

refactor – Boolean
    TRUE if iterative refinement performed. See Section 7.3 and optional parameter reset_ftol.

jmax – Integer
    if st->refactor = TRUE then st->jmax holds the index of the constraint with the maximum violation.

errmax – double
    if st->refactor = TRUE then st->errmax holds the value of the maximum violation.
moved – Boolean
TRUE if some variables moved to their bounds. See the optional parameter reset_ftol.

nmoved – Integer
if st->moved = TRUE then st->nmoved holds the number of variables which were moved to their bounds.

rowerr – Boolean
TRUE if some constraints are not satisfied to within options.ftol.

feasible – Boolean
TRUE when a feasible point has been found.

If comm->sol prt = TRUE then the final result from nag_opt_lp is available and the following members of st are set:

iter – Integer
the number of iterations performed.

n – Integer
the number of variables.

nclin – Integer
the number of linear constraints.

x – double *
x points to the n memory locations holding the final point x.

f – double
the final objective function value or, if x is not feasible, the sum of infeasibilities. If the problem is of type Nag_Fp and x is feasible then f is set to zero.

ax – double *
ax points to the nclin memory locations holding the final values Ax.

state – Integer *
state points to the n+nclin memory locations holding the final status of the variables and general linear constraints. See Section 8.2 for a description of the possible status values.

lambda – double *
lambda points to the n + nclin final values of the Lagrange multipliers.

bl – double *
bl points to the n + nclin lower bound values.

bu – double *
bu points to the n + nclin upper bound values.

endstate – Nag_EndState
the state of termination of nag_opt_lp. Possible values of endstate and their correspondence to the exit value of fail.code are:

<table>
<thead>
<tr>
<th>Value of endstate</th>
<th>Value of fail.code</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nag_Feasible and Nag_Optimal</td>
<td>NE_NOERROR</td>
</tr>
<tr>
<td>Nag_Weakmin</td>
<td>NW_SOLN_NOT_UNIQUE</td>
</tr>
<tr>
<td>Nag_Unbounded</td>
<td>NE_UNBOUNDED</td>
</tr>
<tr>
<td>Nag_Infeasible</td>
<td>NW_NOT_FEASIBLE</td>
</tr>
<tr>
<td>Nag_Too.Many_Iters</td>
<td>NW_TOO_MANY_ITER</td>
</tr>
</tbody>
</table>

The relevant members of the structure comm are:

it_prt – Boolean
will be TRUE when the print function is called with the result of the current iteration.

sol_prt – Boolean
will be TRUE when the print function is called with the final result.
**nag_opt_lp**

new_lm – Boolean
will be **TRUE** when the Lagrange multipliers have been updated.

user – double *
iuser – Integer *
p – Pointer
Pointers for communication of user information. If used they must be allocated memory by
the user either before entry to nag_opt_lp or during a call to print_fun. The type Pointer is
void *.

9. Error Indications and Warnings

**NE_INT_ARG_LT**
On entry, \(n\) must not be less than 1: \(n = \langle\text{value}\rangle\).
On entry, \(nclin\) must not be less than 0: \(nclin = \langle\text{value}\rangle\).

**NE_2_INT_ARG_LT**
On entry, \(tda = \langle\text{value}\rangle\) while \(n = \langle\text{value}\rangle\). These parameters must satisfy \(tda \geq n\).

**NE_OPT_NOT_INIT**
options structure not initialised.

**NE_BAD_PARAM**
On entry parameter options.print_level had an illegal value.
On entry parameter options.prob had an illegal value.
On entry parameter options.start had an illegal value.

**NE_INVALID_INT_RANGE_1**
Value \(\langle\text{value}\rangle\) given to options.max_iter not valid. Correct range is \(max_iter \geq 0\).
Value \(\langle\text{value}\rangle\) given to options.fcheck not valid. Correct range is \(fcheck \geq 1\).

**NE_INVALID_INT_RANGE_2**
Value \(\langle\text{value}\rangle\) given to options.reset_ftol not valid. Correct range is \(0 < \text{reset_ftol} < 1000000\).

**NE_INVALID_REAL_RANGE_FF**
Value \(\langle\text{value}\rangle\) given to options.crash_tol not valid. Correct range is \(0.0 \leq \text{crash_tol} \leq 1.0\).

**NE_INVALID_REAL_RANGE_F**
Value \(\langle\text{value}\rangle\) given to options.ftol not valid. Correct range is \(ftol > 0.0\).
Value \(\langle\text{value}\rangle\) given to options.inf_bound not valid. Correct range is \(inf_bound > 0.0\).
Value \(\langle\text{value}\rangle\) given to options.inf_step not valid. Correct range is \(inf_step > 0.0\).

**NE_CVEC_NULL**
options.prob = \(\langle\text{value}\rangle\) but argument cvec = NULL.

**NE_WARM_START**
options.start = Nag_Warm but pointer options.state = NULL.

**NE_BOUND**
The lower bound for variable \(\langle\text{value}\rangle\) (array element bl[\(\langle\text{value}\rangle\)]) is greater than the upper
bound.

**NE_BOUND_LCON**
The lower bound for linear constraint \(\langle\text{value}\rangle\) (array element bl[\(\langle\text{value}\rangle\)]) is greater than the
upper bound.

**NE_STATE_VAL**
options.state[\(\langle\text{value}\rangle\)] is out of range. state[\(\langle\text{value}\rangle\)] = \(\langle\text{value}\rangle\).

**NE_ALLOC_FAIL**
Memory allocation failed.

If one of the above exits occurs, no values will have been assigned to objf, or to options.ax and
options.lambda. x and options.state will be unchanged.
NW_SOLN_NOT_UNIQUE
Optimal solution is not unique.

\( x \) is a weak local minimum (the projected gradient is negligible, the Lagrange multipliers are optimal but there is a small multiplier). This means that the solution \( x \) is not unique.

NE_UNBOUNDED
Solution appears to be unbounded.

This value of fail.code implies that a step as large as options.inf_step would have to be taken in order to continue the algorithm. This situation can occur only when the problem is of type Nag_LP and at least one variable has no upper or lower bound.

NW_NOT_FEASIBLE
No feasible point was found for the linear constraints.

It was not possible to satisfy all the constraints to within the feasibility tolerance. In this case, the constraint violations at the final \( x \) will reveal a value of the tolerance for which a feasible point will exist – for example, if the feasibility tolerance for each violated constraint exceeds its Residual at the final point. The user should check that there are no constraint redundancies. If the data for the constraints are accurate only to the absolute precision \( \sigma \), the user should ensure that the value of the optional parameter ftol is greater than \( \sigma \). For example, if all elements of \( A \) are of order unity and are accurate only to three decimal places, the optional parameter ftol should be at least \( 10^{-3} \).

NW_TOO_MANY_ITER
The maximum number of iterations, \( \langle \text{value} \rangle \), have been performed.

The value of the optional parameter max_iter may be too small. If the method appears to be making progress (e.g. the objective function is being satisfactorily reduced), increase the value of options.max_iter and rerun nag_opt_lp (possibly using the options.start = Nag_Warm facility to specify the initial working set).

NW_OVERFLOW_WARN
Serious ill-conditioning in the working set after adding constraint \( \langle \text{value} \rangle \). Overflow may occur in subsequent iterations.

If overflow occurs preceded by this warning then serious ill-conditioning has probably occurred in the working set when adding a constraint. It may be possible to avoid the difficulty by increasing the magnitude of the optional parameter ftol and re-running the program. If the message recurs even after this change, the offending linearly dependent constraint \( j \) must be removed from the problem.

NE_NOT_APPEND_FILE
Cannot open file \( \langle \text{string} \rangle \) for appending.

NE_WRITE_ERROR
Error occurred when writing to file \( \langle \text{string} \rangle \).

NE_NOT_CLOSE_FILE
Cannot close file \( \langle \text{string} \rangle \).

10. Further Comments
Sensible scaling of the problem is likely to reduce the number of iterations required and make the problem less sensitive to perturbations in the data, thus improving the condition of the problem. In the absence of better information it is usually sensible to make the Euclidean lengths of each constraint of comparable magnitude. See the Chapter Introduction and Gill et al(1986) for further information and advice.

10.1. Accuracy
nag_opt_lp implements a numerically stable active set strategy and returns solutions that are as accurate as the condition of the problem warrants on the machine.

\[ NP3491/6 \] 3.e04mfc.21
11. References


12. See Also

nag_opt_qp (e04nfc)
nag_opt_init (e04xxc)
nag_opt_read (e04xyc)
nag_opt_free (e04xzc)

13. Example 2

This example is a portfolio investment problem taken from Gill et al (1991b). The objective function to be minimized is

\[-5x_1 - 2x_3 \]

subject to the bounds

\[x_1 \geq -75 \]
\[x_2 \geq -1000 \]
\[x_3 \geq -25 \]

and the general constraints

\[20x_1 + 2x_2 + 100x_3 = 0 \]
\[18x_1 + 3x_2 + 102x_3 \geq -600 \]
\[15x_1 - \frac{1}{2}x_2 - 25x_3 \geq 0 \]
\[-5x_1 + \frac{3}{2}x_2 - 25x_3 \geq -500 \]
\[-5x_1 - \frac{1}{2}x_2 + 75x_3 \geq -1000 \]

The initial point, which is feasible, is

\[x_0 = (10.0, 20.0, 100.0)^T \]

The solution is

\[x^* = (75.0, -250.0, -10.0)^T \]

Three general constraints are active at the solution, the bound constraints are all inactive.

This example shows an optional parameter value assigned directly within the program text while others are read from a data file. The options structure is declared and initialised by nag_opt_init (e04xxc), a value is assigned directly to option inf_bound and nag_opt_qp is then called. On successful return two further options are read from a data file by use of nag_opt_read (e04xyc) and the problem is re-run. The memory freeing function nag_opt_free (e04xzc) is used to free the memory assigned.
to the pointers in the options structure. Users should not use the standard C function free() for this purpose.

13.1. Program Text

```c
static void ex2()
{
    /* This sample linear program (LP) is a portfolio investment problem
     * (see Chapter 7, pp 258--262 of 'Numerical Linear Algebra and
     * The problem involves the rearrangement of a portfolio of three
     * stocks, Glitter, Risky and Trusty, so that the net worth of the
     * investor is maximized.
     * The problem is characterized by the following data:
     * Glitter Risky Trusty
     * 1990 Holdings 75 1000 25
     * 1990 Priceshare($) 20 2 100
     * 2099 Priceshare($) 18 3 102
     * 2099 Dividend 5 0 2
     * The variables x[0], x[1] and x[2] represent the change in each of
     * the three stocks.
     */
    double x[MAXN], cvec[MAXN];
    double a[MAXLIN][MAXN];
    double bl[MAXBND], bu[MAXBND];
    double bigbnd, objf;
    Integer n, tda, nclin;
    Boolean print;
    Nag_E04_Opt options;
    static NagError fail, fail2;
    Vprintf("Example 2: some optional parameters are set.\n");

    fail.print = TRUE;
    fail2.print = TRUE;

    /* Set the actual problem dimensions.
     * n = the number of variables.
     * nclin = the number of general linear constraints (may be 0).
     */
    tda = MAXN;
    n = 3;
    nclin = 5;

    /* Define the value used to denote 'infinite' bounds. */
    bigbnd = 1e+25;

    /* Objective function: maximize 5*X[0] + 2*X[2], or equivalently,
     * minimize -5*X[0] - 2*X[2].
     */
    cvec[0] = -5.0;
    cvec[1] = 0.0;
    cvec[2] = -2.0;

    /* a = the general constraint matrix.
     * bl = the lower bounds on x and A*x.
     * bu = the upper bounds on x and A*x.
     * x = the initial estimate of the solution.
     * A nonnegative amount of stock must be present after rearrangement.
     * For Glitter: x[0] + 75 >= 0.
     * For Risky: x[1] + 1000 >= 0. */
    bl[0] = -75.0;
    bu[0] = bigbnd;
    bl[1] = -1000.0;
    bu[1] = bigbnd;
    ```
/* For Trusty:  \( x[2] + 25 \geq 0 \). */
\begin{align*}
bl[2] &= -25.0; \\
bu[2] &= \text{bigbnd};
\end{align*}

/* The current value of the portfolio must be the same after
* rearrangement, i.e.,
* \( 20 \times (75 + x[0]) + 2 \times (1000 + x[1]) + 100 \times (25 + x[2]) = 6000 \), or
* \( 20 \times x[0] + 2 \times x[1] + 100 \times x[2] = 0 \).
*/
\begin{align*}
a[0][0] &= 20.0; \\
a[0][1] &= 2.0; \\
a[0][2] &= 100.0; \\
b[1][n] &= 0.0; \\
bu[n] &= 0.0;
\end{align*}

/* The value of the portfolio must increase by at least 5 per cent
* at the end of the year, i.e.,
* \( 18 \times (75 + x[0]) + 3 \times (1000 + x[1]) + 102 \times (25 + x[2]) \geq 6300 \), or
* \( 18 \times x[0] + 3 \times x[1] + 102 \times x[2] \geq -600 \).
*/
\begin{align*}
a[1][0] &= 18.0; \\
a[1][1] &= 3.0; \\
a[1][2] &= 102.0; \\
b[1][n + 1] &= -600.0; \\
bu[n + 1] &= \text{bigbnd};
\end{align*}

/* There are three ‘‘balanced portfolio’’ constraints. The value of
* a stock must constitute at least a quarter of the total final
* value of the portfolio. After rearrangement, the value of the
* portfolio after is \( 20 \times (75 + x[0]) + 2 \times (1000 + x[1]) + 100 \times (25 + x[2]) \).
* 
* If Glitter is to constitute at least a quarter of the final
* portfolio, then \( 15 \times x[0] - 0.5 \times x[1] - 25 \times x[2] \geq 0 \).
*/
\begin{align*}
a[2][0] &= 15.0; \\
a[2][1] &= -0.5; \\
a[2][2] &= -25.0; \\
b[1][n + 2] &= 0.0; \\
bu[n + 2] &= \text{bigbnd};
\end{align*}

/* If Risky is to constitute at least a quarter of the final
* portfolio, then \( -5 \times x[0] + 1.5 \times x[1] - 25 \times x[2] \geq -500 \).
*/
\begin{align*}
a[3][0] &= -5.0; \\
a[3][1] &= 1.5; \\
a[3][2] &= -25.0; \\
b[1][n + 3] &= -500.0; \\
bu[n + 3] &= \text{bigbnd};
\end{align*}

/* If Trusty is to constitute at least a quarter of the final
* portfolio, then \( -5 \times x[0] - 0.5 \times x[1] + 75 \times x[2] \geq -1000 \).
*/
\begin{align*}
a[4][0] &= -5.0; \\
a[4][1] &= -0.5; \\
a[4][2] &= 75.0; \\
b[1][n + 4] &= -1000.0; \\
bu[n + 4] &= \text{bigbnd};
\end{align*}

/* Set the initial estimate of the solution.
* This portfolio is infeasible.
*/
\begin{align*}
x[0] &= 10.0; \\
x[1] &= 20.0; \\
x[2] &= 100.0;
\end{align*}

/* Initialise options structure to null values. */
e04xxc(koptions);

/* Set one option */
options.inf_bound = bigbnd;
/* Solve the problem. */
e04mfc(n, nclin, (double *)a, tda, bl, bu, cvec,
x, &objf, &options, NAGCOMM_NULL, &fail);

if (fail.code == NE_NOERROR)
{
    /* Re-solve the problem with some additonal options. */
    Vprintf("Re-solve problem with output of iteration results");
    Vprintf(" suppressed and ftol = 1.0e-10.\n");

    /* Read additional options from a file. */
    fail.print = TRUE;
    print = TRUE;
e04xyc("e04mfc", "stdin", &options, print, "stdout", &fail);

    /* Reset starting point */
x[0] = 0.0;
x[1] = 0.0;
x[2] = 0.0;

    /* Solve the problem again. */
e04mfc(n, nclin, (double *)a, tda, bl, bu, cvec,
x, &objf, &options, NAGCOMM_NULL, &fail);
}

/* Free memory allocated by e04mfc to pointers in options. */
e04xzc(&options, "all", &fail2);
if (fail.code != NE_NOERROR || fail2.code != NE_NOERROR) exit(EXIT_FAILURE);

13.2. Program Data

Following options for e04mfc are read by e04xyc in example 2.

begin e04mfc
    print_level = Nag_Soln /* Print solution only */
    ftol = 1e-10 /* Set feasiblity tolerance */
end

13.3. Program Results

Example 2: some optional parameters are set.

Parameters to e04mfc
--------------------

 Linear constraints........... 5      Number of variables........... 3
 prob.................... Nag_LP  start................... Nag_Cold
 ftol.................... 1.05e-08  reset_ftol.............. 5
 fcheck.................. 50  crash_tol............... 1.00e-02
 inf_bound............... 1.00e+25  inf_step................ 1.00e+25
 max_iter................ 50  machine precision....... 1.11e-16
 optim_tol............... 1.72e-13  min_infeas.............. FALSE
 print_level......... Nag_Soln_Iter
 outfile................. stdout

Memory allocation:
state................... Nag
ax...................... Nag
lambda.................. Nag

Results from e04mfc:
-------------------

<table>
<thead>
<tr>
<th>Itn</th>
<th>Jdel</th>
<th>Jadd</th>
<th>Step</th>
<th>Ninf</th>
<th>Sinf/Obj</th>
<th>Bnd</th>
<th>Lin</th>
<th>Nart</th>
<th>Nrz</th>
<th>Norm Gz</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0e+00</td>
<td>1</td>
<td>1.9369e+02</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1.96e+01</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>A</td>
<td>5.0e-01</td>
<td>0</td>
<td>7.2049e-01</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>4.00e-02</td>
</tr>
</tbody>
</table>
Final solution:

<table>
<thead>
<tr>
<th>Varbl</th>
<th>State</th>
<th>Value</th>
<th>Lower Bound</th>
<th>Upper Bound</th>
<th>Lagr Mult</th>
<th>Residual</th>
</tr>
</thead>
<tbody>
<tr>
<td>V 1</td>
<td>FR</td>
<td>7.5000e+01</td>
<td>-7.5000e+01</td>
<td>None</td>
<td>0.0000e+00</td>
<td>1.5000e+02</td>
</tr>
<tr>
<td>V 2</td>
<td>FR</td>
<td>-2.5000e+02</td>
<td>-1.0000e+03</td>
<td>None</td>
<td>0.0000e+00</td>
<td>7.5000e+02</td>
</tr>
<tr>
<td>V 3</td>
<td>FR</td>
<td>-1.0000e+01</td>
<td>-2.5000e+01</td>
<td>None</td>
<td>0.0000e+00</td>
<td>1.5000e+01</td>
</tr>
</tbody>
</table>

LCon State Value Lower Bound Upper Bound Lagr Mult Residual

L 1 EQ -3.01303e-13 0.0000e+00 0.0000e+00 -1.300e-01 4.780e-13
L 2 FR -4.20000e+02 -6.00000e+02 None 0.0000e+00 1.8000e+02
L 3 FR 1.50000e+03 0.00000e+00 None 0.0000e+00 1.5000e+03
L 4 LL -5.00000e+02 -5.00000e+02 None 2.5000e-01 5.684e-14
L 5 LL -1.00000e+03 -1.00000e+03 None 2.3000e-01 0.0000e+00

Exit after 3 iterations.

Optimal LP solution found.

Final LP objective value = -3.5500000e+02

Re-solve problem with output of iteration results suppressed and ftol = 1.0e-10.

Optional parameter setting for e04mfc.

--------------------------------------
Option file: stdin
print_level set to Nag_Soln
ftol set to 1.00e-10
Parameters to e04mfc
--------------------------------------
Linear constraints........... 5 Number of variables........... 3
prob.................... Nag_LP start................... Nag_Cold
ftol.................... 1.00e-10
fcheck.................. 50
inf_bound............... 1.00e+25
max_iter................ 50
optim_tol............... 1.72e-13
min_infeas............... FALSE
print_level.............. Nag_Soln
outfile................. stdout

Memory allocation:
state.................... Nag
ax........................ Nag
lambda.................... Nag

Final solution:

<table>
<thead>
<tr>
<th>Varbl</th>
<th>State</th>
<th>Value</th>
<th>Lower Bound</th>
<th>Upper Bound</th>
<th>Lagr Mult</th>
<th>Residual</th>
</tr>
</thead>
<tbody>
<tr>
<td>V 1</td>
<td>FR</td>
<td>7.5000e+01</td>
<td>-7.5000e+01</td>
<td>None</td>
<td>0.0000e+00</td>
<td>1.5000e+02</td>
</tr>
<tr>
<td>V 2</td>
<td>FR</td>
<td>-2.5000e+02</td>
<td>-1.0000e+03</td>
<td>None</td>
<td>0.0000e+00</td>
<td>7.5000e+02</td>
</tr>
<tr>
<td>V 3</td>
<td>FR</td>
<td>-1.0000e+01</td>
<td>-2.5000e+01</td>
<td>None</td>
<td>0.0000e+00</td>
<td>1.5000e+01</td>
</tr>
</tbody>
</table>

LCon State Value Lower Bound Upper Bound Lagr Mult Residual

L 1 EQ 4.78019e-13 0.00000e+00 0.00000e+00 -1.3000e-01 4.780e-13
L 2 FR -4.20000e+02 -6.00000e+02 None 0.0000e+00 1.8000e+02
L 3 FR 1.50000e+03 0.00000e+00 None 0.0000e+00 1.5000e+03
L 4 LL -5.00000e+02 -5.00000e+02 None 2.5000e-01 5.684e-14
L 5 LL -1.00000e+03 -1.00000e+03 None 2.3000e-01 3.411e-13

Exit after 2 iterations.
Optimal LP solution found.

Final LP objective value = -3.5500000e+02