nag_ode_ivp_rk_onestep (d02pdc)

1. Purpose
   nag_ode_ivp_rk_onestep (d02pdc) is a one-step function for solving the initial value problem for a first order system of ordinary differential equations using Runge-Kutta methods.

2. Specification
   
   ```c
   #include <nag.h>
   #include <nag02.h>

   void nag_ode_ivp_rk_1step(Integer neq, void (*f)(Integer neq, double t, double y[], double yp[], Nag_User *comm),
                           double *tnow, double ynow[],
                           double ypnow[], Nag_ODE_RK *opt, Nag_User *comm,
                           NagError *fail)
   ```

3. Description
   This function and its associated functions (nag_ode_ivp_rk_setup (d02pvc), nag_ode_ivp_rk_reset_tend (d02pwc), nag_ode_ivp_rk_interp (d02pzc), nag_ode_ivp_rk_errass (d02pzc)) solve the initial value problem for a first order system of ordinary differential equations. The functions, based on Runge-Kutta methods and derived from RKSUITE (Brankin et al, 1991), integrate
   
   \[ y' = f(t, y) \quad \text{given} \quad y(t_0) = y_0 \]
   
   where \( y \) is the vector of \( neq \) solution components and \( t \) is the independent variable.

   This function is designed to be used in complicated tasks when solving systems of ordinary differential equations. You must first call nag_ode_ivp_rk_setup (d02pvc) to specify the problem and how it is to be solved. Thereafter you (repeatedly) call nag_ode_ivp_rk_onestep to take one integration step at a time from \( t \text{start} \) in the direction of \( t \text{end} \) (as specified in nag_ode_ivp_rk_setup (d02pvc)). In this manner nag_ode_ivp_rk_onestep returns an approximation to the solution \( y \text{now} \) and its derivative \( y \text{pnow} \) at successive points \( t \text{now} \). If nag_ode_ivp_rk_onestep encounters some difficulty in taking a step, the integration is not advanced and the routine returns with the same values of \( t \text{now}, y \text{now} \) and \( y \text{pnow} \) as returned on the previous successful step. nag_ode_ivp_rk_onestep tries to advance the integration as far as possible subject to passing the test on the local error and not going past \( t \text{end} \). In the call to nag_ode_ivp_rk_setup (d02pvc) you can specify either the first step size for nag_ode_ivp_rk_onestep to attempt or that it compute automatically an appropriate value. Thereafter nag_ode_ivp_rk_onestep estimates an appropriate step size for its next step. This value and other details of the integration can be obtained after any call to nag_ode_ivp_rk_onestep by examining the contents of the structure \( \text{opt} \), see Section 4. The local error is controlled at every step as specified in nag_ode_ivp_rk_setup (d02pvc). If you wish to assess the true error, you must set \( \text{errass} = \text{Nag_ErrorAssess_on} \) in the call to nag_ode_ivp_rk_setup (d02pvc). This assessment can be obtained after any call to nag_ode_ivp_rk_onestep by a call to the subroutine nag_ode_ivp_rk_errass (d02pzc).

   If you want answers at specific points there are two ways to proceed:

   The more efficient way is to step past the point where a solution is desired, and then call nag_ode_ivp_rk_interp (d02pzc) to get an answer there. Within the span of the current step, you can get all the answers you want at very little cost by repeated calls to nag_ode_ivp_rk_interp (d02pzc). This is very valuable when you want to find where something happens, e.g., where a particular solution component vanishes. You cannot proceed in this way with \( \text{method} = \text{Nag_RK_5_8} \).

   The other way to get an answer at a specific point is to set \( t \text{end} \) to this value and integrate to \( t \text{end} \). nag_ode_ivp_rk_onestep will not step past \( t \text{end} \), so when a step would carry it past, it will reduce the step size so as to produce an answer at \( t \text{end} \) exactly. After getting an answer there (\( t \text{now} = t \text{end} \)), you can reset \( t \text{end} \) to the next point where you want an answer, and repeat. \( t \text{end} \) could be reset by a call to nag_ode_ivp_rk_setup (d02pvc), but you should not do this. You should use
nag_ode_ivp_rk_onestep because it is both easier to use and much more efficient. This way of getting answers at specific points can be used with any of the available methods, but it is the only way with method = Nag_RK_7_8. It can be inefficient. Should this be the case, the code will bring the matter to your attention.

4. Parameters

neq
Input: the number of ordinary differential equations in the system to be solved.
Constraint: neq \geq 1.

f
This function must evaluate the first derivatives \( y'_i \) (that is the functions \( f_i \)) for given values of the arguments \( t, y_i \).

```c
void f (Integer neq, double t, double y[], double yp[], Nag_User *comm)
```

- neq
  Input: the number of differential equations.

- t
  Input: the current value of the independent variable, \( t \).

- y[neq]
  Input: the current values of the dependent variables, \( y_i \) for \( i = 1, 2, \ldots, \text{neq} \).

- yp[neq]
  Output: the values of \( f_i \) for \( i = 1, 2, \ldots, \text{neq} \).

- comm
  Input/Output: pointer to a structure of type Nag_User with the following member:

    - p - Pointer
      Input/Output: The pointer \( \text{comm} \rightarrow p \) should be cast to the required type, e.g. \( \text{struct user} *s = (\text{struct user} *)\text{comm} \rightarrow p \), to obtain the original object’s address with appropriate type. (See the argument \( \text{comm} \) below.)

- tnow
  Output: the value of the independent variable \( t \) at which a solution has been computed.

- ynow[neq]
  Output: an approximation to the solution at \( t_{\text{now}} \). The local error of the step to \( t_{\text{now}} \) was no greater than permitted by the specified tolerances (see nag_ode_ivp_rk_setup (d02pvc)).

- ypnow[neq]
  Output: an approximation to the derivative of the solution at \( t_{\text{now}} \).

- opt
  Input: pointer to a structure of type NagODE_RK as initialised by the setup function nag_ode_ivp_rk_setup (d02pvc).
  Output: the following structure members hold information as follows:

    - totfcn - Integer
      The total number of evaluations of \( f \) used in the primary integration so far; this does not include evaluations of \( f \) for the secondary integration specified by a prior call to nag_ode_ivp_rk_setup (d02pvc) with errass = Nag_ErrorAssess_on.

    - stpcst - Integer
      The cost in terms of number of evaluations of \( f \) of a typical step with the method being used for the integration. The method is specified by the parameter method in a prior call to nag_ode_ivp_rk_setup (d02pvc).

    - waste - double
      The number of attempted steps that failed to meet the local error requirement divided by the total number of steps attempted so far in the integration. A “large” fraction indicates that
the integrator is having trouble with the problem being solved. This can happen when the
problem is “stiff” and also when the solution has discontinuities in a low order derivative.

\textbf{stpsok} - Integer
The number of accepted steps.

\textbf{hnext} - double
The step size the integrator plans to use for the next step.

\textbf{comm}
Input/Output: pointer to a structure of type Nag\_User with the following member:

\textbf{p} - Pointer
Input/Output: The pointer \textit{p}, of type Pointer, allows the user to communicate
information to and from the user-defined function \textit{f()}. An object of the required type
should be declared by the user, e.g. a structure, and its address assigned to the pointer
\textit{p} by means of a cast to Pointer in the calling program, e.g. \textit{comm.p} = (\textbf{Pointer})\&\textit{s}.
The type pointer will be void * with a C compiler that defines void * and char *
otherwise.

\textbf{fail}
The NAG error parameter, see the Essential Introduction to the NAG C Library.

\section{5. Error Indications and Warnings}

\textbf{NE\_PREV\_CALL}
The previous call to a function had resulted in a severe error. You must call
\texttt{nag\_ode\_ivp\_rk\_setup (d02pvc)} to start another problem.

\textbf{NE\_NO\_SETUP}
The setup function \texttt{nag\_ode\_ivp\_rk\_setup (d02pvc)} has not been called.

\textbf{NE\_RK\_INVALID\_CALL}
The function to be called as specified in the setup routine \texttt{nag\_ode\_ivp\_rk\_setup (d02pvc)} was
\texttt{nag\_ode\_ivp\_rk\range (d02pcc)}. However the actual call was made to \texttt{nag\_ode\_ivp\_rk\_onestep}.
This is not permitted.

\textbf{NE\_PREV\_CALL\_INI}
The previous call to the function \texttt{nag\_ode\_ivp\_rk\_onestep} had resulted in a severe error. You
must call \texttt{nag\_ode\_ivp\_rk\_setup (d02pvc)} to start another problem.

\textbf{NE\_NEQ}
The value of \textit{neq} supplied is not the same as that given to the setup function
\texttt{nag\_ode\_ivp\_rk\_setup (d02pvc)}. \textit{neq = (value)} but the value given to \texttt{nag\_ode\_ivp\_rk\_setup}
(d02pvc) was \textit{(value)}.

\textbf{NE\_RK\_PDC\_TEND}
\textit{tend} (\textit{\textasciitilde (value)}) has been reached already. To integrate further with same problem the
function \texttt{nag\_ode\_ivp\_rk\_reset\_tend (d02pvc)} must be called with a new value of \textit{tend}.

\textbf{NE\_RK\_PDC\_STEP}
In order to satisfy the error requirements \texttt{nag\_ode\_ivp\_rk\_onestep} would have to use a step
size of \textit{(value)} at current \textit{t \textasciitilde (value)}. This is too small for the \textit{machine precision}.

\textbf{NW\_RK\_TOO\_MANY}
Approximately \textit{(value)} function evaluations have been used to compute the solution since the
integration started or since this message was last printed.

\textbf{NE\_RK\_PDC\_GLOBAL\_ERROR\_T}
The global error assessment may not be reliable for \textit{t} past \textit{tnow}. \textit{tnow = (value)}.

\textbf{NE\_RK\_PDC\_GLOBAL\_ERROR\_S}
The global error assessment algorithm failed at the start of the integration.
NE_RK_PDC.POINTS
More than 100 output points have been obtained by integrating to tend. They have been sufficiently close to one another that the efficiency of the integration has been degraded. It would probably be (much) more efficient to obtain output by interpolating with nag_ode_ivp_rk_interp (d02pzc) (after changing to method = NagRK.4.5 if you are using method = NagRK.7.8).

NE_STIFF.PROBLEM
The problem appears to be stiff.

NE_MEMORY_FREED
Internally allocated memory has been freed by a call to nag_ode_ivp_rk_free (d02ppc) without a subsequent call to the set up function nag_ode_ivp_rk_setup (d02pvc).

6. Further Comments
If nag_ode_ivp_rk_onestep returns with fail.code = NE_RK_PDC_STEP and the accuracy specified by tol and thres is really required then you should consider whether there is a more fundamental difficulty. For example, the solution may contain a singularity. In such a region the solution components will usually be of a large magnitude. Successive output values of ynow should be monitored with the aim of trapping the solution before the singularity. In any case numerical integration cannot be continued through a singularity, and analytical treatment may be necessary.

If nag_ode_ivp_rk_onestep returns with a non-trivial value of fail.code (i.e., those not related to an invalid call) then performance statistics are available by examining the structure opt (see Section 4). Furthermore if errass was set to Nag_ErrorAssess on then global error assessment is available by a call to the function nag_ode_ivp_rk_errass (d02pzc). The approximate extra number of evaluations of f used is given by $2 \times \text{stpsok} \times \text{stpest}$ for method NagRK.4.5 or NagRK.7.8 and $3 \times \text{stpsok} \times \text{stpest}$ for method = NagRK.2.3.

After a failure with fail.code = NE_RK_PDC_STEP, NE_RK_PDC_GLOBAL_ERROR_T or NE_RK_PDC_GLOBAL_ERROR_S the diagnostic function nag_ode_ivp_rk_errass (d02pzc) may be called only once.

If nag_ode_ivp_rk_onestep returns with fail.code = NE_STIFF.PROBLEM then it is advisable to change to another code more suited to the solution of stiff problems. nag_ode_ivp_rk_onestep will not return with fail.code = NE_STIFF.PROBLEM if the problem is actually stiff but it is estimated that integration can be completed using less function evaluations than already computed.

6.1. Accuracy
The accuracy of integration is determined by the parameters tol and thres in a prior call to nag_ode_ivp_rk_setup (d02pvc). Note that only the local error at each step is controlled by these parameters. The error estimates obtained are not strict bounds but are usually reliable over one step. Over a number of steps the overall error may accumulate in various ways, depending on the properties of the differential system.

6.2. References
Brankin R W, Gladwell I and Shampine L F (1991) RKSUITE: a suite of Runge-Kutta codes for the initial value problem for ODEs SoftReport 91-S1, Department of Mathematics, Southern Methodist University, Dallas, TX 75275, U.S.A.

7. See Also
nag_ode_ivp_rk_setup (d02pvc)
nag_ode_ivp_rk_reset_tend (d02pwc)
nag_ode_ivp_rk_interp (d02pzc)
nag_ode_ivp_rk_errass (d02pzc)
8. Example

We solve the equation

\[ y'' = -y, \quad y(0) = 0, y'(0) = 1 \]

reposed as

\[ y_1' = y_2, \quad y_2' = -y_1 \]

over the range \([0, 2\pi]\) with initial conditions \(y_1(0) = 0.0\) and \(y_2(0) = 1.0\). We use relative error control with threshold values of \(1.0 \times 10^{-8}\) for each solution component and print the solution at each integration step across the range. We use a medium order Runge-Kutta method (\(\text{method} = \text{Nag\_RK\_4\_5}\)) with tolerances \(\text{tol} = 1.0 \times 10^{-4}\) and \(\text{tol} = 1.0 \times 10^{-5}\) in turn so that we may compare the solutions. The value of \(\pi\) is obtained by using X01AAC.

See also the example programs for \(\text{nag\_ode\_ivp\_rk\_reset\_tend}\) (d02pwc) and \(\text{nag\_ode\_ivp\_rk\_interp}\) (d02pxc).

8.1. Program Text

```c
/* nag_ode_ivp_rk_onestep(d02pdc) Example Program */
/* Copyright 1994 Numerical Algorithms Group. */
/* Mark 3, 1994. */
*/
#include <nag.h>
#include <math.h>
#include <stdio.h>
#include <nag_stdlib.h>
#include <nagd02.h>
#include <nagx01.h>

#ifdef NAG_PROTO
static void f(Integer neq, double t1, double y[], double yp[], Nag_User *comm);
#else
static void f();
#endif

#define NEQ 2
#define ZERO 0.0
#define ONE 1.0
#define TWO 2.0
#define FOUR 4.0

main()
{
    Integer neq;
    Nag_RK_method method;
    double hstart, pi, tnow, tend;
    double tol, tstart;
    Integer i;
    double thres[NEQ], ynow[NEQ], ypnow[NEQ], ystart[NEQ];
    Nag_ErrorAssess errass;
    Nag_ODE_RK opt;
    Nag_User comm;

    Vprintf("d02pdc Example Program Results\n");

    /* Set initial conditions and input for d02pvc */
    neq = NEQ;
    pi = X01AAC;
    tstart = ZERO;
    ystart[0] = ZERO;
    ystart[1] = ONE;
    tend = TWO*pi;
```

[NP3275/5/pdf] 3.d02pdc.5
for (i=0; i<neq; i++)
    thres[i] = 1.0e-8;
errass = Nag_ErrorAssess_off;
hstart = ZERO;
method = Nag_RK_4_5;
for (i=1; i<=2; i++)
{
    if (i==1) tol = 1.0e-4;
    if (i==2) tol = 1.0e-5;
    d02pvc(neq, tstart, ystart, tend, tol, thres, method,
           Nag_RK_onestep, errass, hstart, &opt, NAGERR_DEFAULT);
    Vprintf("\nCalculation with tol = %.8le\n", tol);
    Vprintf(" t y1 y2\n");
    Vprintf("%.8f %.8f %.8f\n", tstart, ystart[0], ystart[1]);
    do
        d02pdc(neq, f, &tnow, ynow, ypnow, &opt, &comm,
               NAGERR_DEFAULT);
        Vprintf("%.8f %.8f %.8f\n", tnow, ynow[0], ynow[1]);
        while (tnow<tend);
    Vprintf("\nCost of the integration in evaluations of f is %ld\n", opt.totfcn);
    d02ppc(&opt);
} exit(EXIT_SUCCESS);
#endif
static void f(Integer neq, double t, double y[], double yp[], Nag_User *comm)
#else
    static void f(neq, t, y, yp, comm)
    Integer neq;
    double t;
    double y[], yp[];
    Nag_User *comm;
#endif
{
    yp[0] = y[1];
    yp[1] = -y[0];
}

8.2. Program Data
None.

8.3. Program Results

d02pdc Example Program Results

Calculation with tol = 1.0e-04

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</table>

Cost of the integration in evaluations of f is 78
Calculation with tol = 1.0e-05

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Cost of the integration in evaluations of f is 118