ON THE COMPONENT GROUP OF THE ALGEBRAIC MONODROMY GROUP OF A $K3$ SURFACE

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Abstract. We provide a lower bound for the number of components of the algebraic monodromy group in the situation of a $K3$ surface over a number field k. In the CM case, our bound is sharp. As an application, we describe, in the case of CM, the jump character [CEJ, Definition 2.4.6] entirely in terms of the endomorphism field and the geometric Picard rank.

1. Introduction

1.1. Let X be a $K3$ surface over a number field k. Associated with X, for every prime number l, one has a continuous representation ([SGA5, Exposé VI, Proposition 1.2.5] and [SGA4, Exposé VIII, Théorème 5.2])

$$\varrho_{X,\bar{l}}\colon \operatorname{Gal}(\overline{k}/k) \longrightarrow \operatorname{GL}(T_{\bar{l}}),$$

for $T_{\bar{l}} \subset H^2_{\text{ét}}(X_{\overline{k}}, \overline{\mathbb{Q}}_l(1))$ the transcendental part of the cohomology. The representation $\varrho_{X,\bar{l}}$ gives rise to the *algebraic monodromy group* $G_{X,\bar{l}}$ of X, which is a linear algebraic group over $\overline{\mathbb{Q}}_l$, usually disconnected. It is defined to be the Zariski closure $G_{X,\bar{l}} := \overline{\operatorname{im}(\varrho_{X,\bar{l}})}$ of the image of $\varrho_{X,\bar{l}}$ (cf. Definition 2.5).

For a generic $K3$ surface, one has $G_{X,\bar{l}} = \operatorname{O}(T_{\bar{l}})$. There are, however, cases, in which the jump character $\tau_X\colon \operatorname{Gal}(\overline{k}/k) \to \{\pm 1\}$ (cf. Paragraph 5.2) is trivial [EJ22b, Examples 5.5 and 5.6], which yields that $G_{X,\bar{l}} = \operatorname{SO}(T_{\bar{l}})$. It may happen, too, that the algebraic monodromy group is of positive codimension in $\operatorname{O}(T_{\bar{l}})$, but only when X has a nontrivial endomorphism field $E \supsetneqq \mathbb{Q}$. I.e., when there is real (RM) or complex multiplication (CM). Then, for the neutral component of the algebraic monodromy group, one has

$$G^0_{X,\bar{l}} \cong (\operatorname{C}_E(\operatorname{O}(T_{\bar{l}})))^0\,, \tag{1}$$

due to the work of S. G. Tankeev [Tan90, Tan95], together with Yu. G. Zarhin [Za, Theorem 2.2.1]. We discussed upper bounds for the component group $G_{X,\bar{l}}/G^0_{X,\bar{l}}$ in [EJ22b, Lemma 4.10 and Theorem 4.12].

Remark 1.2. There can, of course, be no hope for a general lower bound. Indeed, the representation $\varrho_{X,\bar{l}}$ induces a homomorphism $\underline{\varrho}_{X,\bar{l}}\colon \operatorname{Gal}(\overline{k}/k) \hookrightarrow G_{X,\bar{l}}/G^0_{X,\bar{l}}$, which is

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surjective. Hence, the component group $G_{X,\overline{l}}/G^0_{X,\overline{l}}$ has a *splitting field* $k' \supseteq k$ satisfying the conditions that $\varrho_{X,\overline{l}}(\mathrm{Gal}(\overline{k}/k')) \subseteq G^0_{X,\overline{l}}$ and that the induced homomorphism $\mathrm{Gal}(k'/k) \to G_{X,\overline{l}}/G^0_{X,\overline{l}}$ is bijective. Then the algebraic monodromy group of the base extension $X_{k'}$ is $G_{X_{k'},\overline{l}} \cong G^0_{X,\overline{l}}$ and, in particular, connected.

Results. There is, however, a lower bound for $K3$ surfaces over number fields that are linearly disjoint to the endomorphism field E.

Theorem 4.5. *Let X be a $K3$ surface over a number field k with endomorphism field E and let l be any prime number.*

a) *If $kE \supseteq k$ is a normal extension then one has a natural surjective homomorphism*

$$G_{X,\overline{l}}/G^0_{X,\overline{l}} \twoheadrightarrow \mathrm{Gal}(kE/k)\,. \tag{2}$$

b) *In any case, $\#(G_{X,\overline{l}}/G^0_{X,\overline{l}})$ is always a multiple of $[kE:k]$.*

The proof shows that the splitting field of $G_{X,\overline{l}}/G^0_{X,\overline{l}}$ contains kE. As the former is normal over k, one might want to conclude that actually there is a natural surjective homomorphism $G_{X,\overline{l}}/G^0_{X,\overline{l}} \twoheadrightarrow \mathrm{Gal}((kE)^{(n)}/k)$, for $(kE)^{(n)}$ the normal closure of kE over k. However, such an improvement is vacuous, at least conjecturally.

Theorem 7.5. *Let X be a $K3$ surface over a subfield K of $\mathbb{C}$ with endomorphism field E. Suppose that the Hodge conjecture holds for $(X \times X)(\mathbb{C})$. Then the composite field KE is normal over K.*

The assumption concerning the Hodge conjecture is fulfilled, for instance, for double covers of $\mathbf{P}^2$ branched over six lines [Sch, Theorem 2].

It is fulfilled, as well, in the CM case [RM, Theorem 5.4], a case in which the methods developed in order to establish Theorem 4.5 are particularly strong.

Theorem 7.7. *Let X be a $K3$ surface over a number field k. Assume that the endomorphism field E of X is a CM field.*

a) *Then there is a natural isomorphism*

$$G_{X,\overline{l}}/G^0_{X,\overline{l}} \stackrel{\cong}{\longrightarrow} \mathrm{Gal}(kE/k)\,.$$

I.e., the splitting field of $G_{X,\overline{l}}/G^0_{X,\overline{l}}$ is exactly kE.

b) *The jump character $\tau_X\colon \mathrm{Gal}(\overline{k}/k) \to \{\pm 1\}$ is*

$$\sigma \mapsto \begin{cases} 1\,, & \text{if } \frac{22-\mathrm{rk}\,\mathrm{Pic}\,X_{\overline{k}}}{[E:\mathbb{Q}]} \text{ is even}\,, \\ \text{sign of the natural permutation action of } \sigma \in \mathrm{Gal}(\overline{k}/k) \text{ on the conjugates } u_1, \overline{u}_1, \ldots, u_d, \overline{u}_d \text{ of a primitive element } u_0 \in E\,, & \text{if } \frac{22-\mathrm{rk}\,\mathrm{Pic}\,X_{\overline{k}}}{[E:\mathbb{Q}]} \text{ is odd}\,. \end{cases}$$

Finally, and again conditional to the Hodge conjecture, the real or complex multiplication on a $K3$ surface over any field $K \subseteq \mathbb{C}$ by some element $u \in E$ may be described by a correspondence, which is a $\mathbb{Q}$-cycle of codimension 2 in $(X \times X)_{\overline{K}}$.

As a digression, we show in Theorem 6.7 that the field of definition of such a correspondence must contain $K(u)$.

Remark 1.3. It is our understanding throughout the article that the algebraic monodomy group is a linear algebraic group over the algebraically closed field $\overline{\mathbb{Q}}_l$. This is a convention just for convenience. One might as well rely on l-adic cohomology instead of the $\bar{l}$-adic theory. Then one ends up with a group scheme $G_{X,l}$ defined over $\mathbb{Q}_l$ that underlies the algebraic monodomy group $G_{X,\bar{l}}$.

The component group is, however, exactly the same, i.e. $G_{X,l}/G^0_{X,l} \cong G_{X,\bar{l}}/G^0_{X,\bar{l}}$. Indeed, by construction, in the irreducible components of $G_{X,l}$, the $\mathbb{Q}_l$-rational points are Zariski dense. Hence, every irreducible component of $G_{X,l}$ is in fact geometrically irreducible [EGA IV, Corollaire 4.5.19.3=$\mathrm{Err}_{\mathrm{IV}}$20].

Conventions and Notation. We follow standard conventions and use notation that is standard in Algebra and Algebraic Geometry. In particular, and perhaps sometimes slightly deviating from this,

i) we often work over a base field, which is usually denoted by k or K.

When $K \supseteq k$ is a field extension, we write $K^{(n)}$ for the *normal closure* of K over k. I.e., for the extension field of k generated by the k-conjugates of K.

For an arbitrary field K, we denote by $\overline{K}$ the algebraic closure.

ii) For $B = (v_1, \ldots, v_m)$ a basis of a K-vector space T, we denote by $\mathrm{M}_B^B(f)$ the matrix representing the K-linear map $f\colon T \to T$. Similarly, for the matrix that represents a K-bilinear form $\langle . \, , . \rangle\colon T \times T \to K$, we write $\mathrm{M}_B(\langle . \, , . \rangle) := (\langle v_i, v_j \rangle)_{1 \leq i,j \leq m}$.

iii) We denote the identity matrix of size m by E_m.

iv) When a surjective homomorphism $\varrho\colon \mathrm{Gal}(k^{(n)}/k) \twoheadrightarrow G$ is given onto a finite group G, then, by the *splitting field* of ϱ (or G), we mean the intermediate field of $k^{(n)}/k$ corresponding to $\ker \varrho$ under the Galois correspondence. The splitting field of G is a Galois extension of k.

v) For a linear algebraic group G over an algebraically closed field, we write G^0 to denote its neutral component.

vi) For a finite dimensional vector space T over an algebraically closed field K, we denote by $\mathrm{GL}(T)$ the linear algebraic group, whose K-rational points are the automorphisms $f\colon T \to T$.

If T is equipped with a non-degenerate symmetric bilinear form $\langle ., . \rangle$, we let $\mathrm{O}(T)$ be the linear algebraic group, whose K-rational points are the orthogonal maps $f\colon T \to T$. I.e. those such that $\langle f(x), f(y) \rangle = \langle x, y \rangle$, for all $x, y \in T$.

As usual, we put $\mathrm{SO}(T) := \mathrm{O}(T)^0$.

vii) We say that a k-algebra E *acts K-linearly* on a K-vector space T or that there is a *K-linear action* of E on T if $K \supseteq k$ and a homomorphism

$$E \hookrightarrow \mathrm{End}_K(T)$$

of k-algebras is provided that respects units.

In this situation, we let $\mathrm{C}_E(\mathrm{O}(T))$ be the *centraliser* of E in $\mathrm{O}(T)$. I.e. the linear algebraic group, whose K-rational points are the orthogonal maps $f\colon T \to T$ that commute with the action of E.

viii) By a $\mathbb{Q}$-*cycle* on a scheme X, we mean a formal $\mathbb{Q}$-linear combination of closed integral subschemes of X, each of which is of the same codimension.

ix) For a Galois extension $k' \supseteq k$ and $\sigma \in \mathrm{Gal}(k'/k)$, we let σ act on $\mathrm{Spec}\, k'$ by the morphism $\sigma\colon \mathrm{Spec}\, k' \to \mathrm{Spec}\, k'$ induced via contravariant functoriality from $\sigma^{-1}\colon k' \to k'$. This defines a left $\mathrm{Gal}(k'/k)$-action on $\mathrm{Spec}\, k$.

Let X be a k-scheme. Slightly abusing notation, we again write σ for the automorphism $X_{k'} = X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k' \stackrel{(\mathrm{id},\sigma)}{\longrightarrow} X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k' = X_{k'}$ of $X_{k'}$ obtained by base change.

2. Technical prerequisites

Independence of l.

Let us recall a few concepts from J.-P. Serre's famous McGill notes [Se68, Ch. I, §2].

Definition 2.1. Let k be a number field and $\varrho\colon \mathrm{Gal}(\overline{k}/k) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ a continuous representation, for a certain prime number l.

a) Then ϱ is called *unramified* at a prime $\mathfrak{p}$ of k if $\varrho(I_\mathfrak{P}) = \{1\}$, for $\mathfrak{P}$ any extension of $\mathfrak{p}$ to $\overline{k}$ and $I_\mathfrak{P}$ the inertia group.

b) The representation ϱ is called *rational* if there is a finite set S of primes of k such that

i) ϱ is unramified at every prime $\mathfrak{p} \notin S$ and

ii) if $\mathfrak{p} \notin S$ then the characteristic polynomial $\chi(\varrho(\mathrm{Frob}_\mathfrak{p})) \in \overline{\mathbb{Q}}_l[T]$ actually has coefficients in $\mathbb{Q}$.

Definition 2.2. Let k be a number field and let $\varrho\colon \mathrm{Gal}(\overline{k}/k) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ and $\varrho'\colon \mathrm{Gal}(\overline{k}/k) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_{l'})$ be two rational continuous representations, for prime numbers l and l'. Then ϱ and ϱ' are said to be *compatible* if there is a finite set S of primes of k such that

i) ϱ and ϱ' are both unramified at every prime $\mathfrak{p} \notin S$ and

ii) the characteristic polynomials $\chi(\varrho(\mathrm{Frob}_\mathfrak{p})), \chi(\varrho'(\mathrm{Frob}_\mathfrak{p})) \in \mathbb{Q}[T]$ coincide for every $\mathfrak{p} \notin S$.

Example 2.3. Let X be a smooth projective scheme over a number field k. Choose integers i and j, as well as a prime number l. Then every element $\sigma \in \mathrm{Gal}(\overline{k}/k)$ induces an automorphism of $X_{\overline{k}}$ and hence, by functoriality, an automorphism $\varrho^{i,j}_{X,\bar{l}}(\sigma) \in \mathrm{GL}(H^i_{\text{ét}}(X_{\overline{k}}, \overline{\mathbb{Q}}_l(j)))$. It is clear that

$$\varrho^{i,j}_{X,\bar{l}}\colon \mathrm{Gal}(\overline{k}/k) \longrightarrow \mathrm{GL}(H^i_{\text{ét}}(X_{\overline{k}}, \overline{\mathbb{Q}}_l(j)))$$

is a representation.

i) One has that $\varrho^{i,j}_{X,\overline{l}}$ is continuous by [SGA5, Exposé VI, Proposition 1.2.5], together with [SGA4, Exposé VIII, Théorème 5.2].

ii) Moreover, $\varrho^{i,j}_{X,\overline{l}}$ is unramified at every prime of k of residue characteristic $\neq l$, at which X has good reduction. This is a consequence of the smooth specialisation theorem [SGA4, Exposé XVI, Corollaire 2.2].

iii) Furthermore, $\varrho^{i,j}_{X,\overline{l}}$ is rational, due to the work of P. Deligne on the Weil conjectures [De74, Théorème (1.6)].

iv) Finally, the representations $\varrho^{i,j}_{X,\overline{l}}$, for X, i, and j fixed, but l running through the set of all primes numbers, are mutually compatible.

Indeed, let $\mathfrak{p}$ be a prime of k of residue characteristic neither l nor l', at which X has good reduction. Then, by the smooth specialisation theorem, $\chi(\varrho^{i,j}_{X,\overline{l}}(\mathrm{Frob}_\mathfrak{p}))$ is the same as the characteristic polynomial of Frob on $H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_\mathfrak{p}}, \overline{\mathbb{Q}}_l(j))$. On the other hand, one has that $\chi(\varrho^{i,j}_{X,\overline{l}'}(\mathrm{Frob}_\mathfrak{p}))$ coincides with the characteristic polynomial of Frob on $H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_\mathfrak{p}}, \overline{\mathbb{Q}}_{l'}(j))$. But the latter two polynomials agree, as a consequence of the Lefschetz trace formula and the Weil conjectures [De74, Théorème (1.6)].

v) If X is connected and $i = 2j = \dim X$ then $H^i_{\text{ét}}(X_{\overline{k}}, \overline{\mathbb{Q}}_l(j))$ carries a non-degenerate symmetric bilinear form, given by the cup product and Poincaré duality [SGA4, Exposé XVIII, Théorème 3.2.5]. One then has $\operatorname{im} \varrho^{i,j}_{X,\overline{l}} \subseteq \mathrm{O}(H^i_{\text{ét}}(X_{\overline{k}}, \overline{\mathbb{Q}}_l(j)))$.

Example 2.4. Let X be a $K3$ surface over a number field k. Choose a prime number l. For c_1 the first Chern class homomorphism, put

$$H_{\overline{l},\mathrm{alg}} := \operatorname{im}(c_1 \otimes_\mathbb{Z} \overline{\mathbb{Q}}_l \colon \operatorname{Pic} X_{\overline{k}} \otimes_\mathbb{Z} \overline{\mathbb{Q}}_l \to H^2_{\text{ét}}(X_{\overline{k}}, \overline{\mathbb{Q}}_l(1)))$$

and $T_{\overline{l}} := (H_{\overline{l},\mathrm{alg}})^\perp$. Then, let

$$\varrho_{X,\overline{l}} \colon \operatorname{Gal}(\overline{k}/k) \longrightarrow \mathrm{O}(T_{\overline{l}}) \subset \mathrm{GL}(T_{\overline{l}})$$

be the restriction of $\varrho^{2,1}_{X,\overline{l}}$ to $T_{\overline{l}}$.

i) As a subrepresentation of $\varrho^{2,1}_{X,\overline{l}}$, one immediately has that $\varrho_{X,\overline{l}}$ is continuous. Moreover, $\varrho_{X,\overline{l}}$ is unramified at every prime, at which $\varrho^{2,1}_{X,\overline{l}}$ is.

ii) Furthermore, $\varrho_{X,\overline{l}}$ is rational. The representations $\varrho_{X,\overline{l}}$, for X fixed, but l running through the set of all primes numbers, are mutually compatible.

Indeed, the characteristic polynomial of the action on $\operatorname{Pic} X_{\overline{k}}$ of any element $\sigma \in \operatorname{Gal}(\overline{k}/k)$ has rational, in fact integral, coefficients, which trivially do not depend on l. The claim directly follows from this.

Definition 2.5. Let k be a number field and let $\varrho\colon \operatorname{Gal}(\overline{k}/k) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ be a continuous representation. Then the Zariski closure $G_\varrho := \overline{\operatorname{im}(\varrho)}$ of the image of ϱ is called the *algebraic monodromy group* of the representation ϱ. It is clear that G_ϱ is a linear algebraic group over $\overline{\mathbb{Q}}_l$.

Remarks 2.6. i) Let l and l' be two prime numbers and let $\varrho\colon \operatorname{Gal}(\overline{k}/k) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ and $\varrho'\colon \operatorname{Gal}(\overline{k}/k) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_{l'})$ be compatible rational representations. Then the algebraic monodromy groups G_ϱ and $G_{\varrho'}$ certainly do not coincide, if only because they are defined over different base fields. Concerning the component groups, there is, however, Theorem 2.7 below, which we are going to make use of.

ii) The representations considered in Example 2.3 give rise to algebraic monodromy groups. Similarly, for a $K3$ surface X over a number field k and a prime number l, there is the algebraic monodromy group $G_{X,\overline{l}} := G_{\varrho_{X,\overline{l}}}$, which is called the *algebraic monodromy group of* X.

Theorem 2.7 (J.-P. Serre)**.** *Let k be a number field and let l and l' be two prime numbers. Suppose that the two rational representations $\varrho\colon \mathrm{Gal}(\overline{k}/k) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ and $\varrho'\colon \mathrm{Gal}(\overline{k}/k) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_{l'})$ are compatible. Then the induced homomorphisms*

$$\underline{\varrho}\colon \mathrm{Gal}(\overline{k}/k) \longrightarrow G_\varrho/G_\varrho^0 \qquad \text{and} \qquad \underline{\varrho}'\colon \mathrm{Gal}(\overline{k}/k) \longrightarrow G_{\varrho'}/G_{\varrho'}^0$$

to the component groups have the same kernel.

Proof. This is shown in [Se81, Lettre du 29/1/1981, p. 18, Théorème]. □

Corollary 2.8. *Let $\varrho_l\colon \mathrm{Gal}(\overline{k}/k) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$, for l running through the set of all primes numbers, be mutually compatible rational representations. Then the splitting field of the induced homomorphism*

$$\underline{\varrho}_l\colon \mathrm{Gal}(\overline{k}/k) \longrightarrow G_{\varrho_l}/G_{\varrho_l}^0$$

to the component group does not depend on l. □

l-adic cohomology versus algebraic de Rham cohomology.

2.9 (The comparison isomorphism)**.** Let $k_\mathfrak{l}$ be a local field of characteristic 0 and residue characteristic $l > 0$. Then $\mathbb{Q}_l \subseteq k_\mathfrak{l}$. We fix inclusions $k_\mathfrak{l} \subseteq \overline{k}_\mathfrak{l} = \overline{\mathbb{Q}}_l \subset \mathbb{C}_l$, for $\mathbb{C}_l$ the completion of $\overline{\mathbb{Q}}_l$, which is again algebraically closed [Ko, Theorem 13]. Moreover, let X be a $K3$ surface over $k_\mathfrak{l}$. Then, as a particular case of p-adic Hodge theory [Fa, Theorem III.4.1], there is a natural $\mathrm{Gal}(\overline{k}_\mathfrak{l}/k_\mathfrak{l})$-equivariant $\mathbb{C}_l$-linear isomorphism

$$\iota_X\colon H^2_{\text{ét}}(X_{\overline{k}_\mathfrak{l}}, \mathbb{C}_l(1)) \stackrel{\cong}{\longrightarrow} \bigoplus_{i=0}^{2} H^i(X_{\overline{k}_\mathfrak{l}}, \Omega^{2-i}_{X_{\overline{k}_\mathfrak{l}}/\overline{k}_\mathfrak{l}}) \otimes_{\overline{k}_\mathfrak{l}} \mathbb{C}_l(i-1)$$

connecting l-adic étale cohomology with algebraic de Rham cohomology.

2.10 (The transcendental parts)**.** The comparison isomorphism ι_X is compatible with the first Chern class homomorphisms. I.e., the diagram

$$\begin{array}{ccccc}
\mathrm{Pic}\, X_{\overline{k}_\mathfrak{l}} & & \xrightarrow{\quad c_1 \quad} & & H^2_{\text{ét}}(X_{\overline{k}_\mathfrak{l}}, \mathbb{C}_l(1)) \\
\Vert & & & & \cong \downarrow \iota_X \\
\mathrm{Pic}\, X_{\overline{k}_\mathfrak{l}} & \xrightarrow{c_1} & H^1(X_{\overline{k}_\mathfrak{l}}, \Omega^1_{X_{\overline{k}_\mathfrak{l}}/\overline{k}_\mathfrak{l}}) & \hookrightarrow & \bigoplus_{i=0}^{2} H^i(X_{\overline{k}_\mathfrak{l}}, \Omega^{2-i}_{X_{\overline{k}_\mathfrak{l}}/\overline{k}_\mathfrak{l}}) \otimes_{\overline{k}_\mathfrak{l}} \mathbb{C}_l(i-1)
\end{array}$$

is commutative. Cf. [Fa, Paragraph III.4.(a)]. Consequently, ι_X induces an isomorphism

$$\iota_X^T\colon T_{\overline{l}}\otimes_{\overline{\mathbb{Q}}_l}\mathbb{C}_l \cong H^0(X_{\overline{k}_{\mathfrak{l}}},\Omega^2_{X_{\overline{k}_{\mathfrak{l}}}/k_{\mathfrak{l}}})\otimes_{k_{\mathfrak{l}}}\mathbb{C}_l(-1)\oplus V_{\overline{k}_{\mathfrak{l}}}\otimes_{\overline{k}_{\mathfrak{l}}}\mathbb{C}_l\oplus H^2(X_{\overline{k}_{\mathfrak{l}}},\mathscr{O}_{X_{\overline{k}_{\mathfrak{l}}}})\otimes_{\overline{k}_{\mathfrak{l}}}\mathbb{C}_l(1)\,,$$

for $V_{\overline{k}_{\mathfrak{l}}} := (c_1(\operatorname{Pic} X_{\overline{k}_{\mathfrak{l}}})\otimes_{\mathbb{Z}}\overline{k}_{\mathfrak{l}})^\perp \subset H^1(X_{\overline{k}_{\mathfrak{l}}},\Omega^1_{X_{\overline{k}_{\mathfrak{l}}}/\overline{k}_{\mathfrak{l}}})$.

2.11 (The number field case)**.** Let k be a number field and X a $K3$ surface over k. For l any prime number, let $\mathfrak{l}$ be a prime of k lying above l.

a) Then, combined with the natural isomorphism $H^2_{\text{ét}}(X_{\overline{k}},\mathbb{C}_l(1)) \stackrel{\cong}{\to} H^2_{\text{ét}}(X_{\overline{k}_{\mathfrak{l}}},\mathbb{C}_l(1))$ [SGA4, Exposé XVI, Corollaire 1.6], $\iota_{X_{k_{\mathfrak{l}}}}$ induces a natural isomorphism

$$\begin{aligned}\iota_{X,\mathfrak{l}}\colon H^2_{\text{ét}}(X_{\overline{k}},\mathbb{C}_l(1)) &\stackrel{\cong}{\longrightarrow} \bigoplus_{i=0}^{2} H^i(X_{\overline{k}_{\mathfrak{l}}},\Omega^{2-i}_{X_{\overline{k}_{\mathfrak{l}}}/\overline{k}_{\mathfrak{l}}})\otimes_{\overline{k}_{\mathfrak{l}}}\mathbb{C}_l(i-1)\\ &\cong \bigoplus_{i=0}^{2} H^i(X_{\overline{k}},\Omega^{2-i}_{X_{\overline{k}}/\overline{k}})\otimes_{\overline{k}}\mathbb{C}_l(i-1)\,,\end{aligned}$$

which is $\mathbb{C}_l$-linear and equivariant with respect to the natural actions of the decomposition group $\operatorname{Gal}(\overline{k}_{\mathfrak{l}}/k_{\mathfrak{l}}) \subset \operatorname{Gal}(\overline{k}/k)$. Restricting to the transcendental part, one obtains a $\operatorname{Gal}(\overline{k}_{\mathfrak{l}}/k_{\mathfrak{l}})$-equivariant $\mathbb{C}_l$-linear isomorphism

$$\iota_{X,\mathfrak{l}}^T\colon T_{\overline{l}}\otimes_{\overline{\mathbb{Q}}_l}\mathbb{C}_l \stackrel{\cong}{\longrightarrow} H^0(X_{\overline{k}},\Omega^2_{X_{\overline{k}}/\overline{k}})\otimes_{\overline{k}}\mathbb{C}_l(-1)\oplus V_{\overline{k}}\otimes_{\overline{k}}\mathbb{C}_l\oplus H^2(X_{\overline{k}},\mathscr{O}_{X_{\overline{k}}})\otimes_{\overline{k}}\mathbb{C}_l(1)\,,$$

for $V_{\overline{k}} := (c_1(\operatorname{Pic} X_{\overline{k}})\otimes_{\mathbb{Z}}\overline{k})^\perp \subset H^1(X_{\overline{k}},\Omega^1_{X_{\overline{k}}/\overline{k}})$.

b) When combining further with the inverse map of the natural isomorphism $H^2_{\text{ét}}(X_{\mathbb{C}},\mathbb{C}_l(1)) \stackrel{\cong}{\leftarrow} H^2_{\text{ét}}(X_{\overline{k}},\mathbb{C}_l(1))$ [SGA4, Exposé XVI, Corollaire 1.6], $\iota_{X,\mathfrak{l}}$ induces a natural isomorphism

$$\begin{aligned}H^2_{\text{ét}}(X_{\mathbb{C}},\mathbb{C}_l(1)) &\stackrel{\cong}{\longrightarrow} \bigoplus_{i=0}^{2} H^i(X_{\overline{k}},\Omega^{2-i}_{X_{\overline{k}}/\overline{k}})\otimes_{\overline{k}}\mathbb{C}_l(i-1)\\ &\cong \bigoplus_{i=0}^{2} H^i(X_{\mathbb{C}},\Omega^{2-i}_{X_{\mathbb{C}}/\mathbb{C}})\otimes_{\mathbb{C}}\mathbb{C}_l(i-1)\,.\end{aligned}$$

Applying the comparison isomorphism $H^2(X(\mathbb{C}),\mathbb{C}_l(1)) \stackrel{\cong}{\leftarrow} H^2_{\text{ét}}(X_{\mathbb{C}},\mathbb{C}_l(1))$ [SGA4, Exposé XI, Théorème 4.4.iii)] to the left and the GAGA isomorphism [Se56, Théorème 1] to the right hand side, this goes over into the usual Hodge decomposition of complex cohomology,

$$H^2(X(\mathbb{C}),\mathbb{C}_l(1)) = \bigoplus_{i=0}^{2} H^i(X(\mathbb{C}),\Omega^{2-i}_{\text{an},X(\mathbb{C})})\otimes_{\mathbb{C}}\mathbb{C}_l(i-1)\,. \tag{3}$$

Note that the Tate twists do not carry any information here, as $\mathbb{C}_l(i) \cong \mathbb{C}_l$, for every $i \in \mathbb{Z}$, and there is no Galois action on complex cohomology.

An elementary descent argument.

Lemma 2.12 (Descent for linear maps)**.** *Let K be a field of characteristic 0 and $L \supseteq K$ an algebraically closed field. For K-vector spaces V and W, let an L-linear map $F\colon V \otimes_K L \to W \otimes_K L$ be given that commutes with the actions of $\operatorname{Aut}_K(L)$. I.e. such that, for every $\sigma \in \operatorname{Aut}_K(L)$, the diagram*

$$\begin{array}{ccc} V \otimes_K L & \xrightarrow{\;F\;} & V \otimes_K L \\ \sigma \big\downarrow & & \big\downarrow \sigma \\ V \otimes_K L & \xrightarrow{\;F\;} & V \otimes_K L \end{array}$$

commutes. Then there is a unique K-linear map $f\colon V \to W$ such that $F = f \otimes_K L$.

Proof. According to [Cl, Corollary 53.b)], one has that $L^{\operatorname{Aut}_K(L)} = K$. Consequently, the only $\operatorname{Aut}_K(L)$-invariant elements in $V \otimes_K L$ are those in V, and analogously for $W \otimes_K L$. By assumption, one has that $F(V)$ is $\operatorname{Aut}_K(L)$-invariant, and therefore $F(V) \subseteq W$. The restriction $f := F|_V\colon V \to W$ is a K-linear map, as V is a K-vector space and F is L-linear. Finally, it is clear that $F = f \otimes_K L$. □

Remark 2.13. This is certainly a particular case of faithfully flat descent [SGA1, Exposé VIII, Lemme 1.4], but Lemma 2.12 suffices for the purposes of this article.

3. The various actions of the endomorphism field

3.1 (Complex cohomology)**.** Let X be a $K3$ surface over a field K that is embeddable into $\mathbb{C}$. Consider the embedding $K \hookrightarrow \mathbb{C}$ as being fixed. The endomorphism field E of X is then defined as follows.

One considers the complex manifold $X(\mathbb{C}) = X_\mathbb{C}(\mathbb{C})$. Then the transcendental part $T := (H_{\text{alg}})^\perp \subset H^2(X(\mathbb{C}), \mathbb{Q})$ of the cohomology is a pure weight-2 $\mathbb{Q}$-Hodge structure [De71, Paragraph 2.1.12 and Définition 2.1.10]. One puts $E := \operatorname{End}_{\text{Hg}}(T)$ to be the endomorphism ring of T in the category of Hodge structures. It is known that E is always a number field [Za, Theorem 1.6.a)], in fact either totally real or a CM field (cf. Paragraph 5.1, below), at least in the realm of $K3$ surfaces.

3.2 (Algebraic de Rham cohomology)**.** Let X be as in 3.1, and let the endomorphism field E of X be as defined above. Then, in particular, one has a $\mathbb{Q}$-linear action of E on $T \subset H^2(X(\mathbb{C}), \mathbb{Q})$. Hence, E acts $\mathbb{C}$-linearly on

$$T \otimes_\mathbb{Q} \mathbb{C} \subset H^2(X(\mathbb{C}), \mathbb{C}) = H^2(X(\mathbb{C}), \mathbb{Q}) \otimes_\mathbb{Q} \mathbb{C}\,,$$

and the action commutes with that of $\operatorname{Aut}(\mathbb{C})$. But

$$\begin{aligned} H^2(X(\mathbb{C}), \mathbb{Q}) \otimes_\mathbb{Q} \mathbb{C} = \bigoplus_{i=0}^{2} H^i(X(\mathbb{C}), \Omega^{2-i}_{\text{an},X(\mathbb{C})}) &\cong \bigoplus_{i=0}^{2} H^i(X_\mathbb{C}, \Omega^{2-i}_{X/\mathbb{C}}) \\ &\cong \bigoplus_{i=0}^{2} H^i(X_{\overline{K}}, \Omega^{2-i}_{X/\overline{K}}) \otimes_{\overline{K}} \mathbb{C}\,, \end{aligned} \tag{4}$$

the actions at least of $\mathrm{Aut}_{\overline{K}}(\mathbb{C})$ agreeing with each other, as both take place only via the right factor. The isomorphism (4) lets the $\mathbb{C}$-linear action of E carry over to the transcendental part within the right hand side. There, it again commutes with the action of $\mathrm{Aut}_{\overline{K}}(\mathbb{C})$.

Thus, Lemma 2.12 yields a $\overline{K}$-linear action of E on

$$H^i(X_{\overline{K}}, \Omega^{2-i}_{X/\overline{K}}),$$

for $i = 0, 2$, as well as on

$$V_{\overline{K}} := (c_1(\mathrm{Pic}\, X_{\overline{\mathbb{C}}}) \otimes_{\mathbb{Z}} \overline{K})^{\perp} = (c_1(\mathrm{Pic}\, X_{\overline{K}}) \otimes_{\mathbb{Z}} \overline{K})^{\perp} \subset H^1(X_{\overline{K}}, \Omega^1_{X/\overline{K}}).$$

3.3 ($\bar{l}$-adic cohomology)**.** Let again X be as in 3.1, and let the endomorphism field E of X be as defined above. Then for any prime number l, the $\mathbb{Q}$-linear action of E on $T \subset H^2(X(\mathbb{C}), \mathbb{Q})$ induces a $\overline{\mathbb{Q}}_l$-linear action of E on

$$\begin{aligned} T_{\bar{l}} = T \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_l(1) \subset H^2(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_l(1) &\cong H^2(X(\mathbb{C}), \overline{\mathbb{Q}}_l(1)) \\ &\cong H^2_{\text{ét}}(X_{\mathbb{C}}, \overline{\mathbb{Q}}_l(1)) \cong H^2_{\text{ét}}(X_{\overline{K}}, \overline{\mathbb{Q}}_l(1)). \end{aligned}$$

Note that the two isomorphisms to the right are canonical [SGA4, Exposé XI, Théorème 4.4.iii) and Exposé XVI, Corollaire 1.6].

Proposition 3.4. *Let X be a $K3$ surface over a number field k. Fix a prime number l and a prime $\mathfrak{l}$ of k lying above l. Then the endomorphism field E of X acts $\overline{\mathbb{Q}}_l$-linearly on $T_{\bar{l}} \subset H^2_{\text{ét}}(X_{\bar{k}}, \overline{\mathbb{Q}}_l(1))$ and in such a way that the* $\mathrm{Gal}(\overline{k}_{\mathfrak{l}}/k_{\mathfrak{l}})$*-equivariant* $\mathbb{C}_l$*-linear isomorphism*

$$\iota^T_{X,\mathfrak{l}} \colon T_{\bar{l}} \otimes_{\overline{\mathbb{Q}}_l} \mathbb{C}_l \xrightarrow{\cong} H^0(X_{\bar{k}}, \Omega^2_{X_{\bar{k}}/\bar{k}}) \otimes_{\bar{k}} \mathbb{C}_l(-1) \oplus V_{\bar{k}} \otimes_{\bar{k}} \mathbb{C}_l \oplus H^2(X_{\bar{k}}, \mathscr{O}_{X_{\bar{k}}}) \otimes_{\bar{k}} \mathbb{C}_l(1)$$

commutes with the actions of E on either side.

Proof. The isomorphism $\iota^T_{X,\mathfrak{l}}$ is just the restriction of the isomorphism

$$\iota_{X,\mathfrak{l}} \colon H^2_{\text{ét}}(X_{\bar{k}}, \mathbb{C}_l(1)) \xrightarrow{\cong} \bigoplus_{i=0}^{2} H^i(X_{\bar{k}}, \Omega^{2-i}_{X_{\bar{k}}/\bar{k}}) \otimes_{\bar{k}} \mathbb{C}_l(i-1),$$

which, according to Paragraph 2.11.b), may be obtained as follows. Start with the Hodge decomposition (3) of $H^2(X(\mathbb{C}), \mathbb{C}_l(1))$ and apply the GAGA isomorphism to the right hand side. Moreover, to the left hand side, apply the canonical comparison isomorphism $H^2_{\text{ét}}(X_{\bar{k}}, \mathbb{C}_l(1)) \xrightarrow{\cong} H^2_{\text{ét}}(X_{\mathbb{C}}, \mathbb{C}_l(1)) \xrightarrow{\cong} H^2(X(\mathbb{C}), \mathbb{C}_l(1))$ [SGA4, Exposé XVI, Corollaire 1.6 and Exposé XI, Théorème 4.4.iii)].

The actions of E on both sides are constructed from one and the same action on $H^2(X(\mathbb{C}), \mathbb{C}_l(1))$ via transport of structure along exactly these isomorphisms. Therefore, they must agree. □

4. The actions of the endomorphism field and the Galois group on algebraic de Rham and l-adic cohomologies—The main results

The general setting–Non-commuting actions.

Convention 4.1. Let X be a $K3$ surface over a subfield K of $\mathbb{C}$. In what follows, we treat the endomorphism field E as being embedded as a subfield into $\overline{K}$ in the following way. The vector space $H^0(X_{\overline{K}}, \Omega^2_{X_{\overline{K}}/\overline{K}})$ is of dimension one, hence the action of E is necessarily given by ${}^{[u]}\omega = \iota(u) \cdot \omega$, for an embedding $\iota\colon E \hookrightarrow \overline{K}$ and arbitrary $u \in E$. We treat E as being a subfield of $\overline{K}$, via ι. Then

$${}^{[u]}\omega = u \cdot \omega\,,$$

for every $\omega \in H^0(X_{\overline{K}}, \Omega^2_{X_{\overline{K}}/\overline{K}})$.

Lemma 4.2 (Non-commuting actions on algebraic de Rham cohomology)**.** *Let X be a $K3$ surface over a subfield K of $\mathbb{C}$ with endomorphism field E.*

a) *Let $\omega \in H^0(X_{\overline{K}}, \Omega^2_{X_{\overline{K}}/\overline{K}})$. Then, for every $\sigma \in \mathrm{Gal}(\overline{K}/K) \subseteq \mathrm{Aut}_{\mathbb{Q}}(\overline{K})$ and each $u \in E$, one has*

$${}^{\sigma\circ[u]\circ\sigma^{-1}}\omega = {}^{[\sigma^{-1}(u)]}\omega\,.$$

b) *Suppose, in particular, that $\sigma \in \mathrm{Gal}(\overline{K}/K) \subseteq \mathrm{Aut}_{\mathbb{Q}}(\overline{K})$ and $u \in E$ are of the kind that $\sigma(u) \neq u$. Then the actions of σ and $[u]$ on $H^0(X_{\overline{K}}, \Omega^2_{X_{\overline{K}}/\overline{K}})$ do not commute with each other.*

Proof. a) One has ${}^{[u]}\omega = u \cdot \omega$, for every $u \in E$ and $\omega \in H^0(X_{\overline{K}}, \Omega^2_{X_{\overline{K}}/\overline{K}})$. On the other hand, the action of σ is provided by pull-back,

$${}^{\sigma}\omega = \sigma^*\omega\,.$$

Consequently, one finds

$${}^{\sigma\circ[u]\circ\sigma^{-1}}\omega = {}^{\sigma}({}^{[u]}({}^{\sigma^{-1}}\omega)) = \sigma^*(u \cdot (\sigma^{-1})^*\omega) = \sigma^{-1}(u) \cdot \sigma^*((\sigma^{-1})^*\omega) = \sigma^{-1}(u) \cdot \omega\,,$$

as claimed. Recall here that $\sigma\colon X_{\overline{K}} \to X_{\overline{K}}$ is induced by the field automorphism $\sigma^{-1}\colon \overline{K} \to \overline{K}$.

b) As the assumption implies that $u \neq \sigma^{-1}(u)$, part a) shows that the actions of $\sigma\circ[u]\circ\sigma^{-1}$ and $[u]$ on $H^0(X_{\overline{K}}, \Omega^2_{X_{\overline{K}}/\overline{K}})$ do not coincide. This is equivalent to the assertion. □

A lower bound for the component group of the algebraic monodromy group.

Proposition 4.3. *Let X be a $K3$ surface over a number field k with endomorphism field E. Moreover, let $\sigma \in \mathrm{Gal}(\overline{k}/k)\backslash\mathrm{Gal}(\overline{k}/kE)$ be arbitrary. Then one has*

$$\varrho_{X,\bar{l}}(\sigma) \notin G^0_{X,\bar{l}}\,,$$

for any prime number l.

Proof. *First step.* Adjusting the automorphism σ. Fixing a prime number l.

Let $k' \supseteq k$ denote the splitting field of $G_{X,\bar{l}}/G^0_{X,\bar{l}}$, which is a normal extension and, according to Corollary 2.8, independent of l. Moreover, consider the normal closure

$(kE)^{(n)}$ of kE as an extension field of k. Then $(kE)^{(n)} \supseteq k$ and hence $k'(kE)^{(n)} \supseteq k$, too, are normal extensions. It follows that the assumption, as well as the assertion, depend only on the restriction $\sigma|_{k'(kE)^{(n)}} \in \mathrm{Gal}(k'(kE)^{(n)}/k)$.

Since $k'(kE)^{(n)} \supseteq k$ is an extension of number fields, the Chebotarev density theorem provides a prime $\mathfrak{L}$ of $k'(kE)^{(n)}$ that is unramified over k such that $\mathrm{Frob}^{\mathfrak{L}}_{k'(kE)^{(n)}/k} = \sigma|_{k'(kE)^{(n)}}$. We denote the prime of k lying under $\mathfrak{L}$ by $\mathfrak{l}$. Let us choose a further extension $\mathfrak{M}$ of $\mathfrak{L}$ to $\overline{k}$ and put

$$\widetilde{\sigma} := \mathrm{Frob}^{\mathfrak{M}}_{\overline{k}/k} \in \mathrm{Gal}(\overline{k_{\mathfrak{l}}}/k_{\mathfrak{l}}) \hookrightarrow \mathrm{Gal}(\overline{k}/k)\,.$$

Then $\widetilde{\sigma}|_{k'(kE)^{(n)}} = \sigma|_{k'(kE)^{(n)}} \in \mathrm{Gal}(k'(kE)^{(n)}/k)$, so that it suffices to show the assertion for $\widetilde{\sigma}$, instead of σ. Note at this point that the choice of $\mathfrak{M}$ distinguishes a particular embedding $\mathrm{Gal}(\overline{k_{\mathfrak{l}}}/k_{\mathfrak{l}}) \hookrightarrow \mathrm{Gal}(\overline{k}/k)$.

Let l be the prime number lying under the prime $\mathfrak{l}$ of k. In the steps below, l-adic cohomology is always meant to be for this value of l.

Second step. The actions on algebraic de Rham cohomology.

By assumption, there is an element $u_0 \in E$ such that $\widetilde{\sigma}(u_0) \neq u_0$. Then Lemma 4.2.b) yields that the actions of $\widetilde{\sigma}$ and $[u_0]$ on

$$H^0(X_{\overline{k}_{\mathfrak{l}}}, \Omega^2_{X_{\overline{k}_{\mathfrak{l}}}/\overline{k}_{\mathfrak{l}}})$$

do not commute with each other. Consequently, the actions of $\widetilde{\sigma}\circ[u_0]\circ\widetilde{\sigma}^{-1}$ and $[u_0]$ on

$$H^0(X_{\overline{k}_{\mathfrak{l}}}, \Omega^2_{X_{\overline{k}_{\mathfrak{l}}}/\overline{k}_{\mathfrak{l}}})\otimes_{\overline{k}_{\mathfrak{l}}}\mathbb{C}_l(-1) \oplus V_{\overline{k}_{\mathfrak{l}}}\otimes_{\overline{k}_{\mathfrak{l}}}\mathbb{C}_l \oplus H^2(X_{\overline{k}_{\mathfrak{l}}}, \mathscr{O}_{X_{\overline{k}_{\mathfrak{l}}}})\otimes_{\overline{k}_{\mathfrak{l}}}\mathbb{C}_l(1)\,,$$

which are both $\mathbb{C}_l$-linear maps, must be different, as well.

Third step. The transfer to l-adic cohomology.

According to Proposition 3.4, the actions of $\widetilde{\sigma}\circ[u_0]\circ\widetilde{\sigma}^{-1}$ and $[u_0]$ on

$$T_{\overline{l}}\otimes_{\overline{\mathbb{Q}}_l}\mathbb{C}_l\,,$$

which are again both $\mathbb{C}_l$-linear maps, do not coincide either. This implies that the underlying $\overline{\mathbb{Q}}_l$-linear endomorphisms of $T_{\overline{l}}$ already differ from each other. In other words, the action of $\widetilde{\sigma} \in \mathrm{Gal}(\overline{k_{\mathfrak{l}}}/k_{\mathfrak{l}})$ on $T_{\overline{l}}$ does not commute with that of $u_0 \in E$.

I.e., $\varrho_{X,\overline{l}}(\widetilde{\sigma}) \notin \mathrm{C}_E(\mathrm{O}(T_{\overline{l}}))$. In particular, one certainly has $\varrho_{X,\overline{l}}(\widetilde{\sigma}) \notin (\mathrm{C}_E(\mathrm{O}(T_{\overline{l}})))^0$, which, by the work of S. G. Tankeev [Tan90, Tan95] and Yu. G. Zarhin [Za, Theorem 2.2.1], is equivalent to $\varrho_{X,\overline{l}}(\widetilde{\sigma}) \notin G^0_{X,\overline{l}}$. Cf. formula (1) from the introduction.

Fourth step. Conclusion.

Let $l' \neq l$ be an arbitrary prime number. Then one has $\varrho_{X,\overline{l}'}(\widetilde{\sigma}) \notin G^0_{X,\overline{l}'}$, too, by Theorem 2.7, which completes the proof. □

Corollary 4.4. *Let X be a $K3$ surface over a number field k with endomorphism field E. Then, for any prime number l, the splitting field of $G_{X,\overline{l}}/G^0_{X,\overline{l}}$ contains kE as a subfield.* □

Theorem 4.5. *Let X be a $K3$ surface over a number field k with endomorphism field E and let l be any prime number.*

a) *There is a natural surjective homomorphism $G_{X,\bar{l}}/G^0_{X,\bar{l}} \twoheadrightarrow \mathrm{Gal}((kE)^{(n)}/k)$. If $kE \supseteq k$ is a normal extension then one has a natural surjective homomorphism*

$$G_{X,\bar{l}}/G^0_{X,\bar{l}} \twoheadrightarrow \mathrm{Gal}(kE/k)\,. \tag{5}$$

b) *In particular, $\#(G_{X,\bar{l}}/G^0_{X,\bar{l}})$ is always a multiple of $[kE:k]$.*

Proof. a) Write $k' \supseteq k$ for the splitting field of $G_{X,\bar{l}}/G^0_{X,\bar{l}}$. Then k' is normal over k and one has a natural isomorphism

$$G_{X,\bar{l}}/G^0_{X,\bar{l}} \stackrel{\cong}{\longrightarrow} \mathrm{Gal}(k'/k)\,.$$

Corollary 4.4 implies that $k' \supseteq kE$. Therefore, it follows that $k' \supseteq (kE)^{(n)}$ holds, too, which implies the first claim. The second claim follows immediately.

b) is a direct consequence of a). □

Remark 4.6. At least under the assumption of the Hodge conjecture, $kE \supseteq k$ is always a normal extension of fields. Cf. Theorem 7.5.

5. The fundamental lemma on the CM case

5.1. Recall that a number field E is said to be a *CM field* if E is a totally imaginary quadratic extension of a totally real field. Let E be a CM field and let $u_0 \in E$ be a primitive element. Then $\mathrm{Gal}(E^{(n)}/\mathbb{Q})$ acts transitively on the conjugates $u_1, \overline{u}_1, \ldots, u_d, \overline{u}_d$ of u_0, thereby preserving the obvious system of blocks of size 2. Moreover, the complex conjugation just flips each block, and is therefore a central element in $\mathrm{Gal}(E^{(n)}/\mathbb{Q})$ [Sh, Proposition 5.11].

A $K3$ surface X over a number field k is said to have *complex multiplication (CM)* if the endomorphism field E is a CM field. In this situation, let $E_0 \subset E$ be the maximal totally real subfield, $d := [E_0 : \mathbb{Q}]$, and $r := \dim_{\overline{\mathbb{Q}}} T$. Then, for every prime number l, the action of u_0 splits $T_{\bar{l}}$ into eigenspaces $T_{\bar{l}} = T_{l,u_1} \oplus T_{l,\overline{u}_1} \oplus \cdots \oplus T_{l,u_d} \oplus T_{l,\overline{u}_d}$, which come in pairs corresponding to the conjugates of u_0. Each eigenspace is of dimension $\frac{r}{2d}$. Indeed, in $T \subset H^2(X(\mathbb{C}), \mathbb{Q})$, the eigenspaces are mutually conjugate, and hence of the same dimension. Thus, the claim follows from Proposition 3.4.

5.2. Let, as before, X be a $K3$ surface over a number field k. Then the *jump character* of X (cf. [CEJ, Definition 2.4.6]) is the homomorphism $\tau_X\colon \mathrm{Gal}(\overline{k}/k) \to \{\pm 1\}$ given by the natural action of $\mathrm{Gal}(\overline{k}/k)$ on $\bigwedge^{\max} T_{\bar{l}}$.

Remarks 5.3. i) The jump character is independent of l [CEJ, Proposition 2.2.1.b)].

ii) The name "jump character" has not been chosen at random. In fact, suppose that $\mathrm{rk}\,\mathrm{Pic}\, X_{\overline{k}}$ is even and let $\mathfrak{p}$ be a prime of k, at which X has good reduction and such that the jump character of X evaluates to (-1) at $\mathrm{Frob}_\mathfrak{p}$. Then $\mathrm{rk}\,\mathrm{Pic}\, X_{\overline{\mathbb{F}}_\mathfrak{p}} \geq \mathrm{rk}\,\mathrm{Pic}\, X_{\overline{k}} + 2$ [CEJ, Proposition 2.4.2].

5.4. Contrary to the generic and RM cases, in the CM case, one has that (5) is always an isomorphism. I.e., that the splitting field of $G_{X,\overline{l}}/G^0_{X,\overline{l}}$ is exactly kE. Cf. Theorem 7.7.a). As a consequence, it is possible in the CM case to describe the jump character entirely in terms of the endomorphism field and the Picard rank, cf. Theorem 7.7.b).

The goal of this section is to provide the necessary preparations. The complete argument will be given in Section 7 and is, unfortunately, somewhat indirect. It makes use of the Hodge conjecture, which is known to be true in the case of CM $K3$ surfaces.

Proposition 5.5. *Let X be a $K3$ surface over a number field k having CM by an endomorphism field E. Choose a prime number l.*

a) *Then each of the eigenspaces $T_{l,u_1}, T_{l,\overline{u}_1}, \ldots, T_{l,u_d}, T_{l,\overline{u}_d}$ is isotropic. One has $\mathrm{C}_E(\mathrm{O}(T_{\overline{l}})) \cong [\mathrm{GL}_{r/2d}(\overline{\mathbb{Q}}_l)]^d$. In particular, $G^0_{X,\overline{l}} \cong \mathrm{C}_E(\mathrm{O}(T_{\overline{l}}))$.*

b) *There is a natural faithful permutation action of $G_{X,\overline{l}}/G^0_{X,\overline{l}}$ on the set of eigenspaces $\{T_{l,u_1}, T_{l,\overline{u}_1}, \ldots, T_{l,u_d}, T_{l,\overline{u}_d}\}$ that preserves the obvious system of blocks of size 2,*

$$G_{X,\overline{l}}/G^0_{X,\overline{l}} \hookrightarrow (\mathbb{Z}/2\mathbb{Z})^d \rtimes S_d \subseteq S_{2d} \,.$$

Proof. a) For two eigenvectors, $\langle v, v'\rangle \neq 0$ is possible only when the corresponding eigenvalues u, u' are complex conjugates. Indeed one has

$$u\cdot\langle v, v'\rangle = \langle {}^{[u_0]}v, v'\rangle = \langle v, {}^{[\overline{u}_0]}v'\rangle = \overline{u}'\cdot\langle v, v'\rangle \,,$$

due to [Za, Theorem 1.5.1]. In particular, the first claim is shown.

Moreover, the cup product pairing on $T_{\overline{l}}$ is nondegenerate, so there exist bases $B_{u_1}, B_{\overline{u}_1}, \ldots, B_{u_d}, B_{\overline{u}_d}$ of T_{l,u_1}, $T_{l,\overline{u}_1}$, $\ldots$, T_{l,u_d}, and $T_{l,\overline{u}_d}$, respectively, such that, for $B := B_{u_1} \cup B_{\overline{u}_1} \cup \cdots \cup B_{u_d} \cup B_{\overline{u}_d}$, the matrix $\mathrm{M}_B(\langle .,. \rangle)$ is block diagonal consisting of d diagonal blocks of type

$$\left(\begin{smallmatrix} 0 & E_{r/2d} \\ E_{r/2d} & 0 \end{smallmatrix}\right)$$

and zeroes, otherwise.

Furthermore, an endomorphism $s \in \mathrm{End}(T_{\overline{l}})$ commutes with the action of E if and only if it maps the eigenspaces to themselves. I.e., when

$$\mathrm{M}^B_B(s) = \begin{pmatrix} G_1^+ & 0 & \ldots & 0 & 0 \\ 0 & G_1^- & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & G_d^+ & 0 \\ 0 & 0 & \ldots & 0 & G_d^- \end{pmatrix} \,,$$

for suitable matrices $G_1^+, G_1^-, \ldots, G_d^+, G_d^-$. On the other hand, a direct calculation shows that s is orthogonal if and only if $G_i^- = ((G_i^+)^t)^{-1}$, for each i. Consequently,

$$\mathrm{C}_E(\mathrm{O}(T_{\overline{l}})) = \left\{ s \in \mathrm{End}(T_{\overline{l}}) \;\middle|\; \mathrm{M}^B_B(s) = \begin{pmatrix} G_1 & 0 & \ldots & 0 & 0 \\ 0 & (G_1^t)^{-1} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & G_d & 0 \\ 0 & 0 & \ldots & 0 & (G_d^t)^{-1} \end{pmatrix}, \text{ for some } G_1, \ldots, G_d \right\}$$

$$\cong [\mathrm{GL}_{r/2d}(\overline{\mathbb{Q}}_l)]^d\,,$$

as claimed. In particular, $\mathrm{C}_E(\mathrm{O}(T_{\overline{l}}))$ is connected, which implies the final assertion in view of (1).

b) As the neutral component of an algebraic group is always normal, one clearly has

$$G_{X,\overline{l}} \subseteq \mathrm{N}_{\mathrm{O}(T_{\overline{l}})}(G^0_{X,\overline{l}}) \cong \mathrm{N}_{\mathrm{O}(T_{\overline{l}})}(\mathrm{C}_E(\mathrm{O}(T_{\overline{l}})))\,.$$

Moreover, if $s \in \mathrm{O}(T_{\overline{l}}) \subset \mathrm{End}(T_{\overline{l}})$ normalises $G^0_{X,\overline{l}} \cong \mathrm{C}_E(\mathrm{O}(T_{\overline{l}}))$ then each of the subvector spaces $s(T_{l,u_1}), s(T_{l,\overline{u}_1}), \ldots, s(T_{l,u_d}), s(T_{l,\overline{u}_d})$ must be invariant under the action of $\mathrm{C}_E(\mathrm{O}(T_{\overline{l}}))$. Since there are no such spaces other than $T_{l,u_1}, T_{l,\overline{u}_1}, \ldots, T_{l,u_d}, T_{l,\overline{u}_d}$, the endomorphism s necessarily permutes them, thereby defining a permutation action

$$G_{X,\overline{l}} \longrightarrow \mathrm{Sym}(\{T_{l,u_1}, T_{l,\overline{u}_1}, \ldots, T_{l,u_d}, T_{l,\overline{u}_d}\}) \cong S_{2d}\,. \tag{6}$$

Furthermore, $s \in \mathrm{O}(T_{\overline{l}}) \subset \mathrm{End}(T_{\overline{l}})$ commutes with E if and only if s maps every eigenspace to itself, so the kernel of (6) is exactly $\mathrm{C}_E(\mathrm{O}(T_{\overline{l}})) \cong G^0_{X,\overline{l}}$, which implies the first assertion. The preservation of the block system is obvious. □

Lemma 5.6 (The fundamental lemma on the CM case)**.** *Let X be a $K3$ surface over a number field k having CM by an endomorphism field E. For some prime number l, suppose that*

$$^{\sigma\circ[u]\circ\sigma^{-1}}v = {}^{[\sigma^{-1}(u)]}v\,, \tag{7}$$

for every $v \in T_{\overline{l}}$, $\sigma \in \mathrm{Gal}(\overline{k}/k)$, and $u \in E$.

a) *Then there is a natural isomorphism*

$$G_{X,\overline{l}}/G^0_{X,\overline{l}} \stackrel{\cong}{\longrightarrow} \mathrm{Gal}(kE/k)\,.$$

I.e., the splitting field of $G_{X,\overline{l}}/G^0_{X,\overline{l}}$ is exactly kE.

b) *The jump character $\tau_X\colon\ \mathrm{Gal}(\overline{k}/k) \to \{\pm 1\}$ is*

$$\sigma \mapsto \begin{cases} 1\,, & \text{if } \frac{22-\mathrm{rk}\,\mathrm{Pic}\,X_{\overline{k}}}{[E:\mathbb{Q}]} \text{ is even}\,,\\ \text{sign of the natural permutation action of } \sigma \in \mathrm{Gal}(\overline{k}/k) \text{ on the conjugates } u_1, \overline{u}_1, \ldots, u_d, \overline{u}_d \text{ of a primitive element } u_0 \in E\,, & \text{if } \frac{22-\mathrm{rk}\,\mathrm{Pic}\,X_{\overline{k}}}{[E:\mathbb{Q}]} \text{ is odd}\,. \end{cases}$$

Proof. a) The assertion means that $\varrho_{X,\overline{l}}(\sigma) \in G^0_{X,\overline{l}}$ if and only if $\sigma \in \mathrm{Gal}(\overline{k}/kE)$, which is equivalent to

$$\varrho_{X,\overline{l}}(\sigma) \in G^0_{X,\overline{l}} \iff \sigma|_E = \mathrm{id}_E\,. \tag{8}$$

The implication "$\Longrightarrow$" of (8) is true, due to Theorem 4.5.a). In order to show "$\Longleftarrow$", note that, for $\sigma|_E = \mathrm{id}_E$, assumption (7) says that the action of σ on $T_{\overline{l}}$ commutes with that of E. But this is sufficient to imply $\varrho_{X,\overline{l}}(\sigma) \in G^0_{X,\overline{l}}$ in the CM case, due to Proposition 5.5.a). For the statement concerning the splitting field of $G_{X,\overline{l}}/G^0_{X,\overline{l}}$, recall part iv) of the Conventions and Notation subsection.

b) Assumption (7) yields $^{\sigma\circ[\sigma(u_0)]}v = {}^{[u_0]}({}^{\sigma}v)$, for every $\sigma \in \mathrm{Gal}(\overline{k}/k)$ and $v \in T_{\overline{l}}$. Hence, for an eigenvector $v \in T_{l,\nu}$, one has $^{\sigma}v \in T_{l,\sigma(\nu)}$. In other words, the action of $\mathrm{Gal}(\overline{k}/k)$ permutes the eigenspaces $T_{l,u_1}, T_{l,\overline{u}_1}, \ldots, T_{l,u_d}, T_{l,\overline{u}_d}$ according to the natural action of $\mathrm{Gal}(\overline{k}/k)$ on $u_1, \overline{u}_1, \ldots, u_d, \overline{u}_d$, as elements of $\overline{k}$.

On the other hand, by definition, the jump character maps each $\sigma \in \mathrm{Gal}(\overline{k}/k)$ to $\det \varrho_{X,\overline{l}}(\sigma)$. As det is locally constant, this depends only on the class of $\varrho_{X,\overline{l}}(\sigma) \in G_{X,\overline{l}}$ in the factor group $G_{X,\overline{l}}/G^0_{X,\overline{l}} \hookrightarrow (\mathbb{Z}/2\mathbb{Z})^d \rtimes S_d$. By what was just shown, this class is given directly by the natural action of σ on $u_1, \overline{u}_1, \ldots, u_d, \overline{u}_d$.

We claim, more generally, that, for every $A \in \mathrm{N}_{\mathrm{O}(T_{\overline{l}})}(\mathrm{C}_E(\mathrm{O}(T_{\overline{l}})))$, one has

$$\det A = (\mathrm{sgn}(\pi_A, a_A))^{r/2d} ,$$

where (π_A, a_A) denotes the class of A in

$$\mathrm{N}_{\mathrm{O}(T_{\overline{l}})}(\mathrm{C}_E(\mathrm{O}(T_{\overline{l}})))/\,\mathrm{C}_E(\mathrm{O}(T_{\overline{l}})) \cong (\mathbb{Z}/2\mathbb{Z})^d \rtimes S_d \subseteq S_{2d} \, .$$

Note here that $\frac{r}{2d} = \frac{22 - \mathrm{rk}\,\mathrm{Pic}\,X_{\overline{k}}}{[E:\mathbb{Q}]}$.

For this, choose bases $B_1 = \{v_{1,1}, \ldots, v_{1,r/2d}\}$, $\ldots$, $B_d = \{v_{d,1}, \ldots, v_{d,r/2d}\}$ of $T_{l,u_1}, \ldots, T_{l,u_d}$, respectively, and equip $T_{l,\overline{u}_i}$ with the basis $B_i^* = \{v^*_{1,1}, \ldots, v^*_{1,r/2d}\}$, dual to B_i, for $i = 1, \ldots, d$. Moreover, for every $(\pi, a) \in (\mathbb{Z}/2\mathbb{Z})^d \rtimes S_d \subseteq S_{2d}$, let $M_{(\pi,a)}$ be the $\mathbb{Q}_l$-linear map that sends $v_{i,j}$ to $v_{\pi(i),j}$ or $v^*_{\pi(i),j}$, depending on whether $a_i = 0$ or 1, and $v^*_{i,j}$ to $v^*_{\pi(i),j}$ or $v_{\pi(i),j}$, accordingly. Then, by construction, $M_{(\pi,a)}$ is an orthogonal map, hence $M_{(\pi,a)} \in \mathrm{N}_{\mathrm{O}(T_{\overline{l}})}(\mathrm{C}_E(\mathrm{O}(T_{\overline{l}})))$. On the other hand, $M_{(\pi,a)}$ is the $\mathbb{Q}_l$-linear map corresponding to a permutation matrix, for the permutation given by $\frac{r}{2d}$ disjoint copies of (π, a). Therefore, $\det M_{(\pi,a)} = (\mathrm{sgn}(\pi, a))^{r/2d}$. Finally, for $A \in \mathrm{N}_{\mathrm{O}(T_{\overline{l}})}(\mathrm{C}_E(\mathrm{O}(T_{\overline{l}})))$ arbitrary, $M_{(\pi_A,a_A)}$ belongs to the same component of the algebraic group $\mathrm{N}_{\mathrm{O}(T_{\overline{l}})}(\mathrm{C}_E(\mathrm{O}(T_{\overline{l}})))$ as A. This finally shows that $\det A = \det M_{(\pi_A,a_A)} = (\mathrm{sgn}(\pi_A, a_A))^{r/2d}$, as claimed. □

Remark 5.7. Supposedly, assumption (7) is always true, even in the non-CM cases. Our argument, however, relies on the Hodge conjecture and is therefore postponed to Section 7. Cf. Lemma 7.6.b), below.

6. A digression: the field of definition of the conjectural correspondence describing real or complex multiplication

6.1. Let X be a $K3$ surface over a field $K \subseteq \mathbb{C}$. Then one has the direct decomposition $H^2(X(\mathbb{C}), \mathbb{Q}) = H_{\mathrm{alg}} \oplus T$ of pure weight-2 $\mathbb{Q}$-Hodge structures.

Moreover, let E be the endomorphism field of X. Then every $u \in E$ defines a morphism $[u]\colon T \to T$ of Hodge structures. There exists a (non-unique) extension $H^2(X(\mathbb{C}), \mathbb{Q}) \to H^2(X(\mathbb{C}), \mathbb{Q})$ of $[u]$ that is still a morphism of Hodge structures. Let us fix one such extension, which, by abuse of notation, we again denote by $[u]$.

Then $[u]\colon H^2(X(\mathbb{C}), \mathbb{Q}) \to H^2(X(\mathbb{C}), \mathbb{Q})$ is a $\mathbb{Q}$-linear map and the induced $\mathbb{C}$-linear map $H^2(X(\mathbb{C}), \mathbb{C}) \to H^2(X(\mathbb{C}), \mathbb{C})$, which respects the Hodge decomposition $H^2(X(\mathbb{C}), \mathbb{C}) = \bigoplus_{i=0}^2 H^i(X(\mathbb{C}), \Omega^{2-i}_{\mathrm{an},X(\mathbb{C})})$, clearly commutes with the action of $\mathrm{Aut}_{\overline{K}}(\mathbb{C})$ on both sides. Therefore, Lemma 2.12 implies that $[u]$ descends from a $\mathbb{C}$-linear endomorphism of $H^i(X(\mathbb{C}), \Omega^{2-i}_{\mathrm{an},X(\mathbb{C})}) = H^i(X_{\mathbb{C}}, \Omega^{2-i}_{X_{\mathbb{C}}/\mathbb{C}})$ to a $\overline{K}$-linear endomorphism of $H^i(X_{\overline{K}}, \Omega^{2-i}_{X_{\overline{K}}/\overline{K}})$, for $i = 0, 1, 2$. Cf. Paragraph 3.2.

Definition 6.2. Let X be a $K3$ surface with endomorphism field E over a field $K \subseteq \mathbb{C}$, and let $u \in E$ be arbitrary.

a) For a basis $(v_1^{\mathrm{an}}, \ldots, v_{22}^{\mathrm{an}})$ of $H^2(X(\mathbb{C}), \mathbb{Q})$ and $((v_1^{\mathrm{an}})^*, \ldots, (v_{22}^{\mathrm{an}})^*)$ the dual basis, the *analytic Casimir element* Δ_u^{an} associated with u is given by

$$\begin{aligned}\Delta_u^{\mathrm{an}} := (v_1^{\mathrm{an}})^* \otimes {}^{[u]}v_1^{\mathrm{an}} + \cdots + (v_{22}^{\mathrm{an}})^* \otimes {}^{[u]}v_{22}^{\mathrm{an}} &\in H^2(X(\mathbb{C}), \mathbb{Q})^* \otimes_{\mathbb{Q}} H^2(X(\mathbb{C}), \mathbb{Q}) \\ &\cong H^2(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} H^2(X(\mathbb{C}), \mathbb{Q}) \\ &\subset H^4((X \times X)(\mathbb{C}), \mathbb{Q}) .\end{aligned}$$

b) For a basis $(v_1, \ldots, v_{22})$ of $\bigoplus_{i=0}^2 H^i(X_{\overline{K}}, \Omega^{2-i}_{X_{\overline{K}}/\overline{K}})$ and $(v_1^*, \ldots, v_{22}^*)$ the dual basis, the *algebraic Casimir element* Δ_u associated with u is given by

$$\begin{aligned}\Delta_u := v_1^* \otimes {}^{[u]}v_1 + \cdots + v_{22}^* \otimes {}^{[u]}v_{22} &\in \bigoplus_{i=0}^{2} H^i(X_{\overline{K}}, \Omega^{2-i}_{X_{\overline{K}}/\overline{K}})^* \otimes_{\overline{K}} \bigoplus_{i=0}^{2} H^i(X_{\overline{K}}, \Omega^{2-i}_{X_{\overline{K}}/\overline{K}}) \qquad (9) \\ &\cong \bigoplus_{i=0}^{2} H^i(X_{\overline{K}}, \Omega^{2-i}_{X_{\overline{K}}/\overline{K}}) \otimes_{\overline{K}} \bigoplus_{i=0}^{2} H^i(X_{\overline{K}}, \Omega^{2-i}_{X_{\overline{K}}/\overline{K}}) \\ &\subset \bigoplus_{i=0}^{4} H^i((X \times X)_{\overline{K}}, \Omega^{4-i}_{(X \times X)_{\overline{K}}/\overline{K}}) ,\end{aligned}$$

Here,

$$[u]\colon \bigoplus_{i=0}^{2} H^i(X_{\overline{K}}, \Omega^{2-i}_{X_{\overline{K}}/\overline{K}}) \longrightarrow \bigoplus_{i=0}^{2} H^i(X_{\overline{K}}, \Omega^{2-i}_{X_{\overline{K}}/\overline{K}}) , \qquad v \mapsto {}^{[u]}v$$

is the endomorphism described above.

Remarks 6.3. i) The Casimir elements Δ_u and Δ_u^{an} do not depend on the bases chosen. Cf. [Hu, Section 6.2], where this fact is shown in a rather different context.

ii) In particular, the image of Δ_u^{an} in

$$H^4((X \times X)(\mathbb{C}), \mathbb{C}) \cong \bigoplus_{i=0}^{4} H^i((X \times X)_{\mathbb{C}}, \Omega^{4-i}_{(X \times X)_{\mathbb{C}}/\mathbb{C}})$$

under change of coefficients may be constructed in the same way from any basis of

$$H^2(X(\mathbb{C}), \mathbb{C}) = \bigoplus_{i=0}^{2} H^i(X(\mathbb{C}), \Omega^{2-i}_{\mathrm{an}, X(\mathbb{C})}) = \bigoplus_{i=0}^{2} H^i(X_{\mathbb{C}}, \Omega^{2-i}_{X_{\mathbb{C}}/\mathbb{C}}) .$$

Therefore, it coincides with the image of Δ_u in $\bigoplus_{i=0}^4 H^i((X \times X)_{\mathbb{C}}, \Omega^{4-i}_{(X \times X)_{\mathbb{C}}/\mathbb{C}})$ under base change.

Lemma 6.4. *Let X be a $K3$ surface with endomorphism field E over a field $K \subseteq \mathbb{C}$, and let $u \in E$ be arbitrary. Then, for any $\sigma \in \mathrm{Gal}(\overline{K}/K)$ and any basis $(v_1, \ldots, v_{22})$*

of $\bigoplus_{i=0}^{2} H^i(X_{\overline{K}}, \Omega^{2-i}_{X_{\overline{K}}/\overline{K}})$*, one has*

$$^{\sigma}(\Delta_u) = v_1^* \otimes {}^{(\sigma\circ[u]\circ\sigma^{-1})}v_1 + \cdots + v_{22}^* \otimes {}^{(\sigma\circ[u]\circ\sigma^{-1})}v_{22} \,.$$

Proof. Recall that Δ_u is independent of the basis chosen. Instead of $(v_1, \ldots, v_{22})$, another basis is provided by $({}^{\sigma^{-1}}v_1, \ldots, {}^{\sigma^{-1}}v_{22})$. Moreover, as σ^{-1} acts by an orthogonal map, the dual basis is then simply $({}^{\sigma^{-1}}(v_1^*), \ldots, {}^{\sigma^{-1}}(v_{22}^*))$. Consequently, one has

$$\Delta_u = {}^{\sigma^{-1}}(v_1^*) \otimes {}^{[u]}({}^{\sigma^{-1}}v_1) + \cdots + {}^{\sigma^{-1}}(v_{22}^*) \otimes {}^{[u]}({}^{\sigma^{-1}}v_{22}) \,,$$

which yields that

$$\begin{aligned} {}^{\sigma}(\Delta_u) &= v_1^* \otimes {}^{\sigma}({}^{[u]}({}^{\sigma^{-1}}v_1)) + \cdots + v_{22}^* \otimes {}^{\sigma}({}^{[u]}({}^{\sigma^{-1}}v_{22})) \\ &= v_1^* \otimes {}^{(\sigma\circ[u]\circ\sigma^{-1})}v_1 + \cdots + v_{22}^* \otimes {}^{(\sigma\circ[u]\circ\sigma^{-1})}v_{22} \,, \end{aligned}$$

as claimed. □

Proposition 6.5. *Let X be a $K3$ surface with endomorphism field E over a field $K \subseteq \mathbb{C}$, and let $u \in E$ be arbitrary. Suppose that the Hodge conjecture* [De06] *holds for $(X\times X)(\mathbb{C})$.*

a) *Then there exists a $\mathbb{Q}$-cycle C_u of codimension 2 in $(X\times X)_{\overline{K}}$ of the kind that*

$$\mathrm{cl}(C_u) = \Delta_u \,. \tag{10}$$

b) *There is a finite extension field $K' \supseteq K$ such that the $\mathbb{Q}$-cycle C_u can be defined over K'.*

Definition 6.6. The $\mathbb{Q}$-cycle C_u is usually called a *correspondence* describing the action of $u \in E$ on $X_{\overline{K}}$.

Proof of Proposition 6.5. a) As $[u]\colon H^2(X(\mathbb{C}), \mathbb{Q}) \to H^2(X(\mathbb{C}), \mathbb{Q})$ is a morphism of Hodge structures, the image in $H^4((X\times X)(\mathbb{C}), \mathbb{C})$ under change of coefficients of the analytic Casimir element $\Delta_u^{\mathrm{an}} \in H^4((X\times X)(\mathbb{C}), \mathbb{Q})$ is pure of Hodge type $(2,2)$. Therefore, the Hodge conjecture yields a $\mathbb{Q}$-cycle $C_u^{\mathbb{C}}$ of codimension 2 in

$$(X\times X)(\mathbb{C}) = X(\mathbb{C})\times X(\mathbb{C})$$

corresponding to the analytic Casimir element Δ_u^{an}. I.e., of the kind that

$$\mathrm{cl}^{\mathrm{an}}(C_u^{\mathbb{C}}) = \Delta_u^{\mathrm{an}} \,. \tag{11}$$

According to the GAGA principle [Se56, Proposition 13], one knows that $C_u^{\mathbb{C}}$ is automatically a complex algebraic $\mathbb{Q}$-cycle on $(X\times X)_{\mathbb{C}}$. Furthermore, the analytic cycle class map $\mathrm{cl}^{\mathrm{an}}$ is compatible with the algebraic cycle class map cl.

The cycle class map is compatible with base change from $\overline{K}$ to $\mathbb{C}$, too. Cf. [SP, tag 0FWC]. There is, however, no need for $C_u^{\mathbb{C}}$ to descend to $(X\times X)_{\overline{K}}$. On the other hand, algebraic equivalence is finer than homological equivalence [Fu, Proposition 19.1.1]. Hence, one may replace $C_u^{\mathbb{C}}$ by any algebraically equivalent $\mathbb{Q}$-cycle C_u on $(X\times X)_{\mathbb{C}}$, without affecting (11).

Write

$$C_u^{\mathbb{C}} = r_1(C_1) + \cdots + r_m(C_m) \,,$$

for $r_1, \ldots, r_m \in \mathbb{Q}$ and $C_1, \ldots, C_m \subset (X \times X)_\mathbb{C}$ closed subschemes of codimension 2 being reduced and irreducible. We shall treat each component individually, replacing them by algebraically equivalent ones that descend to $\overline{K}$.

To be concrete, let us first choose a projective embedding $X \times X \hookrightarrow \mathbf{P}^N_K$. Then, for each i, there is the Hilbert scheme $H_i := H_{P_{C_i}, X \times X}$, which is a projective K-scheme [FGA, Exposé 221, Théorème 3.2], that parameterises closed subschemes of $X \times X$ having the same Hilbert polynomial P_{C_i} as C_i. The Hilbert scheme H_i comes equipped with a universal family $\pi_i \colon \mathscr{C}_i \hookrightarrow X \times X \times H_i \stackrel{\mathrm{pr}}{\twoheadrightarrow} H_i$ that is a projective and flat morphism of schemes. Moreover, by [EGA IV, Théorème 12.2.4.(viii)], the locus $H_i^0 \subset H_i$, above which the fibres of π are reduced and irreducible, is open.

The subscheme $C_i \subset (X \times X)_\mathbb{C}$ gives rise to a $\mathbb{C}$-rational point z_i on H_i^0, according to the definition of the Hilbert scheme. In particular, one sees that $H_i^0 \neq \emptyset$. Let $(H_i^0)' \subseteq H_i^0$ be the irreducible component of H_i^0 containing z_i. Then $(H_i^0)'$ has a $\overline{K}$-rational point $z_i \in (H_i^0)'(\overline{K})$ by a weak version of Hilbert's Nullstellensatz [Ei, Corollary 13.12.i]. The corresponding fibre $\widetilde{C}_i := (\mathscr{C}_i \times_{H_i} \overline{\{z_i\}})_\mathbb{C}$ is obviously algebraically equivalent to C_i, cf. [Fu, Example 10.3.2], and descends to $\overline{K}$. This completes the proof of a).

b) immediately follows from a). $\square$

Theorem 6.7 (Lower bound for the field of definition of a correspondence)**.** *Let X be a $K3$ surface with endomorphism field E over a field $K \subseteq \mathbb{C}$. Suppose that the Hodge conjecture holds for $(X \times X)(\mathbb{C})$. For some $u \in E$, let C_u be a correspondence describing the action of u on $X_{\overline{K}}$ and let $K' \supseteq K$ be an extension field over which C_u can be defined.*

Then $K' \supseteq E'$, for $E' := K(u)$ the subfield of E generated by u.

Proof. As C_u can be defined over K', one has ${}^\sigma(C_u) = C_u$, for every $\sigma \in \mathrm{Gal}(\overline{K}/K')$. Since the cycle map is $\mathrm{Gal}(\overline{K}/K')$-equivariant, this yields

$$ {}^\sigma(\Delta_u) = {}^\sigma(\mathrm{cl}(C_u)) = \mathrm{cl}({}^\sigma(C_u)) = \mathrm{cl}(C_u) = \Delta_u \,. \tag{12}$$

At this point, Lemma 6.4 shows in view of (9) that ${}^{(\sigma \circ [u] \circ \sigma^{-1})}v_i = {}^{[u]}v_i$, for $i = 1, \ldots, 22$. Thus, the actions of $\sigma \circ [u] \circ \sigma^{-1}$ and $[u]$ on

$$ \bigoplus_{i=0}^{2} H^i(X_{\overline{K}}, \Omega^{2-i}_{X_{\overline{K}}/\overline{K}}) \,, $$

which are both $\overline{K}$-linear maps, coincide. In other words, the action of σ commutes with that of $[u]$.

From this, Lemma 4.2.b) immediately yields that $\sigma(u) = u$. As $\sigma \in \mathrm{Gal}(\overline{K}/K')$ is arbitrary, this is possible only for $u \in K'$. I.e., one has $K(u) \subseteq K'$, as required. $\square$

7. Results relying on the Hodge conjecture

Generalities.

7.1. Let X be a $K3$ surface over a field $K \subseteq \mathbb{C}$. Then, as usual, a $\mathbb{Q}$-cycle C of codimension 2 in $(X \times X)_{\overline{K}}$ defines a homomorphism on algebraic de Rham cohomology [Kl, Section 3],

$$\gamma_C\colon \bigoplus_{i=0}^{2} H^i(X_{\overline{K}}, \Omega^{2-i}_{X_{\overline{K}}/\overline{K}}) \longrightarrow \bigoplus_{i=0}^{2} H^i(X_{\overline{K}}, \Omega^{2-i}_{X_{\overline{K}}/\overline{K}}), \quad v \mapsto \mathrm{pr}_{2*}(\mathrm{pr}_1^*(v) \cup \mathrm{cl}(C)).$$

If $\mathrm{cl}(C) = v_1^* \otimes w_1 + \cdots + v_{22}^* \otimes w_{22}$, for $(v_1, \ldots, v_{22})$ a basis of $\bigoplus\limits_{i=0}^{2} H^i(X_{\overline{K}}, \Omega^{2-i}_{X_{\overline{K}}/\overline{K}})$ and $(v_1^*, \ldots, v_{22}^*)$ the dual basis, then

$$\gamma_C(v_i) = w_i\,, \tag{13}$$

for $i = 1, \ldots, 22$.

7.2. Let E be the endomorphism field of X. For each $u \in E$, we extend $[u]\colon T \to T$ to a $\mathbb{Q}$-linear map

$$[u]\colon H^2(X(\mathbb{C}), \mathbb{Q}) \to H^2(X(\mathbb{C}), \mathbb{Q})$$

by mapping H_{alg} identically to 0. This is a particular case of the extensions considered in the section above.

Suppose, in addition, that the Hodge conjecture holds for $(X \times X)(\mathbb{C})$. Then, according to Proposition 6.5.a), there is a $\mathbb{Q}$-cycle C_u of codimension 2 in $(X \times X)_{\overline{K}}$ satisfying the condition that $\mathrm{cl}(C_u) = \Delta_u$. By (13), for every $u \in \bigoplus_{i=0}^{2} H^i(X_{\overline{K}}, \Omega^{2-i}_{X_{\overline{K}}/\overline{K}})$, one has

$$\gamma_{C_u}(v) = {}^{[u]}v\,.$$

Furthermore, for arbitrary $\sigma \in \mathrm{Gal}(\overline{K}/K)$, let us consider the $\mathbb{Q}$-cycle ${}^{\sigma}(C_u)$, which is again of codimension 2 in $(X \times X)_{\overline{K}}$. One clearly has $\mathrm{cl}({}^{\sigma}(C_u)) = {}^{\sigma}(\Delta_u)$. Thus, from Lemma 6.4, together with (13), one concludes that

$$\gamma_{{}^{\sigma}(C_u)}(v) = {}^{\sigma \circ [u] \circ \sigma^{-1}}v\,, \tag{14}$$

for every $u \in \bigoplus\limits_{i=0}^{2} H^i(X_{\overline{K}}, \Omega^{2-i}_{X_{\overline{K}}/\overline{K}})$.

7.3. Analogously to the above, a $\mathbb{Q}$-cycle C of codimension 2 on $(X \times X)_{\overline{K}}$ also defines a homomorphism on l-adic cohomology,

$$\gamma_{C,l}\colon H^2_{\text{ét}}(X_{\overline{K}}, \overline{\mathbb{Q}}_l(1)) \longrightarrow H^2_{\text{ét}}(X_{\overline{K}}, \overline{\mathbb{Q}}_l(1))\,, \quad v \mapsto \mathrm{pr}_{2*}(\mathrm{pr}_1^*(v) \cup \mathrm{cl}_l(C))\,.$$

Suppose that $k = K$ is a number field. Then there is the comparison isomorphism $\iota_{X,\mathfrak{l}}$, which is compatible with the cycle map, cup products, and the Künneth decomposition [Fa, Theorem II.3.1, cf. Paragraph 2.11]. So one has a commutative diagram

$$\begin{array}{ccc} H^2_{\text{ét}}(X_{\overline{k}}, \mathbb{C}_l(1)) & \xrightarrow{\gamma_{C,l}\otimes\text{id}} & H^2_{\text{ét}}(X_{\overline{k}}, \mathbb{C}_l(1)) \\ \cong\downarrow \iota_{X,\mathfrak{l}} & & \cong\downarrow \iota_{X,\mathfrak{l}} \\ \bigoplus\limits_{i=0}^{2} H^i(X_{\overline{k}}, \Omega^{2-i}_{X_{\overline{k}}/\overline{k}}) \otimes_{\overline{k}} \mathbb{C}_l(i-1) & \xrightarrow{\gamma_C\otimes\text{id}} & \bigoplus\limits_{i=0}^{2} H^i(X_{\overline{k}}, \Omega^{2-i}_{X_{\overline{k}}/\overline{k}}) \otimes_{\overline{k}} \mathbb{C}_l(i-1)\,. \end{array}$$

Therefore, Proposition 3.4 shows that

$$\gamma_{C_u,l}(v) = {}^{[u]}v\,, \tag{15}$$

for every $v \in T_{\overline{l}}$, too. Moreover, the l-adic cycle map cl_l, as well as cup products and the Künneth decomposition, are compatible with the action of $\text{Gal}(\overline{k}/k)$. Thus, (15) implies that $\gamma_{^\sigma(C_u),l}(^\sigma v) = {}^\sigma({}^{[u]}v)$, which is equivalent to

$$\gamma_{^\sigma(C_u),l}(v) = {}^{\sigma\circ[u]\circ\sigma^{-1}}v\,. \tag{16}$$

Lemma 7.4. *Let X be a $K3$ surface with endomorphism field E over a field $K \subseteq \mathbb{C}$ and C a $\mathbb{Q}$-cycle of codimension 2 in $(X\times X)_{\overline{K}}$. Furthermore, let some nonzero $\omega \in H^0(X_{\overline{K}}, \Omega^2_{X_{\overline{K}}/\overline{K}})$ be given.*

a) *Then $\gamma_C(\omega) = u\cdot\omega$, for some scalar factor u that is necessarily an element of E.*

b) *Suppose, in particular, that $\gamma_C(\omega) = 0$. Then the homomorphism γ_C is identically zero on $H^i(X_{\overline{K}}, \Omega^{2-i}_{X_{\overline{K}}/\overline{K}})$, for $i = 0, 2$, as well as on $V_{\overline{K}}$.*

Proof. The homomorphism $\gamma_C^{\text{an}}\colon H^2(X(\mathbb{C}), \mathbb{Q}) \to H^2(X(\mathbb{C}), \mathbb{Q})$ on complex cohomology induced by γ_C may be described by $\gamma_C^{\text{an}}\colon v \mapsto \text{pr}_{2*}(\text{pr}_1^*(v) \cup \text{cl}^{\text{an}}(C_\mathbb{C}))$, for $C_\mathbb{C}$ the complexification of C. Therefore, γ_C^{an} is a morphism of Hodge structures [GH, Section 2.5], which implies that it must map the transcendental part $T \subset H^2(X(\mathbb{C}), \mathbb{Q})$ to itself. Clearly, $\gamma_C^{\text{an}}|_T\colon T \to T$ is an endomorphism of the Hodge structure T. I.e., $\gamma_C^{\text{an}}|_T$ and hence

$$\gamma_C|_{\bigoplus\limits_{i=0,2} H^i(X_{\overline{K}},\Omega^{2-i}_{X_{\overline{K}}/\overline{K}})\oplus V_{\overline{K}}}\colon \bigoplus_{i=0,2} H^i(X_{\overline{K}}, \Omega^{2-i}_{X_{\overline{K}}/\overline{K}}) \oplus V_{\overline{K}} \longrightarrow \bigoplus_{i=0,2} H^i(X_{\overline{K}}, \Omega^{2-i}_{X_{\overline{K}}/\overline{K}}) \oplus V_{\overline{K}}$$

itself, too, must be given by an element $u \in E$.

a) In particular, one has that $\gamma_C|_{H^0(X_{\overline{K}},\Omega^2_{X_{\overline{K}}/\overline{K}})}$ is simply the map $\gamma_C\colon \omega \mapsto u\cdot\omega$, which proves the assertion.

b) Here, the assumption yields that $u = 0$. Therefore, $\gamma_C|_{\bigoplus\limits_{i=0,2} H^i(X_{\overline{K}},\Omega^{2-i}_{X_{\overline{K}}/\overline{K}})\oplus V_{\overline{K}}} = 0$, as claimed. ∎

Normality of the endomorphism field.

The result below is well-known in the context of CM elliptic curves [Sh, Proposition 5.17.(3)].

Theorem 7.5 (Normality of the endomorphism field)**.** *Let X be a $K3$ surface with endomorphism field E over a field $K \subseteq \mathbb{C}$. Suppose that the Hodge conjecture holds for $(X \times X)(\mathbb{C})$. Then the composite field KE is normal over K.*

Proof. Let $u_0 \in E$ be a primitive element. As the Hodge conjecture is assumed to hold for $(X \times X)(\mathbb{C})$, by Proposition 6.5.a), there is a $\mathbb{Q}$-cycle C_{u_0} of codimension 2 in $(X \times X)_{\overline{K}}$ satisfying $\mathrm{cl}(C_{u_0}) = \Delta_{u_0}$. For an arbitrary $\sigma \in \mathrm{Gal}(\overline{K}/K)$, let us consider the $\mathbb{Q}$-cycle ${}^{\sigma}(C_{u_0})$, which is again of codimension 2 in $(X \times X)_{\overline{K}}$. Concerning the homomorphism on algebraic de Rham cohomology defined by ${}^{\sigma}(C_{u_0})$, one knows from formula (14) that

$$\gamma_{{}^{\sigma}(C_{u_0})}(v) = {}^{\sigma \circ [u_0] \circ \sigma^{-1}} v \, ,$$

for every $u \in \bigoplus_{i=0}^{2} H^i(X_{\overline{K}}, \Omega^{2-i}_{X_{\overline{K}}/\overline{K}})$.

For $v \in H^0(X_{\overline{K}}, \Omega^2_{X_{\overline{K}}/\overline{K}})$, Lemma 4.2.a) computes the right hand side explicitly. One has

$$\gamma_{{}^{\sigma}(C_{u_0})}(v) = \sigma^{-1}(u_0) \cdot v \, . \tag{17}$$

At this point, from Lemma 7.4.a), one sees that $\sigma^{-1}(u_0) \in E \subseteq KE$ is enforced. Since $u_0 \in KE$ is a primitive element relative to K and $\sigma \in \mathrm{Gal}(\overline{K}/K)$ is arbitrary, this shows that KE is normal over K, which completes the proof. □

The splitting field and the jump character in the CM case.

Lemma 7.6 (Commuting actions)**.** *Let X be a $K3$ surface with endomorphism field E over a field $K \subseteq \mathbb{C}$. Suppose that the Hodge conjecture holds for $(X \times X)(\mathbb{C})$.*
a) *Then, for every $u \in E$ and $\sigma \in \mathrm{Gal}(\overline{K}/K) \subseteq \mathrm{Aut}_{\mathbb{Q}}(\overline{K})$, one has*

$${}^{\sigma \circ [u] \circ \sigma^{-1}} \eta = {}^{[\sigma^{-1}(u)]} \eta \, ,$$

for arbitrary $\eta \in \bigoplus_{i=0}^{2} H^i(X_{\overline{K}}, \Omega^{j}_{X_{\overline{K}}/\overline{K}})$.
b) *Suppose, in particular, that $k = K$ is a number field. Let l be any prime number and let $v \in T_{\overline{l}}$ be arbitrary. Then, for every $\sigma \in \mathrm{Gal}(\overline{k}/k)$ and each $u \in E$,*

$${}^{\sigma \circ [u] \circ \sigma^{-1}} v = {}^{[\sigma^{-1}(u)]} v \, .$$

Proof. a) For the $\mathbb{Q}$-cycle $D := {}^{\sigma}(C_u) - C_{[\sigma^{-1}(u)]}$ on $(X \times X)_{\overline{K}}$, one has

$$\gamma_D(\eta) = {}^{\sigma \circ [u] \circ \sigma^{-1}} \eta - {}^{[\sigma^{-1}(u)]} \eta \, ,$$

for every $\eta \in \bigoplus_{i=0}^{2} H^i(X_{\overline{K}}, \Omega^{2-i}_{X_{\overline{K}}/\overline{K}})$. Therefore, Lemma 4.2 shows that γ_D vanishes on $H^0(X_{\overline{K}}, \Omega^2_{X_{\overline{K}}/\overline{K}})$. Hence, according to Lemma 7.4.b), γ_D is the zero map on the whole of

$$\bigoplus_{i=0,2} H^i(X_{\overline{K}}, \Omega^{2-i}_{X_{\overline{K}}/\overline{K}}) \oplus V_{\overline{K}} \, .$$

As the actions of u and $\sigma^{-1}(u)$ are assumed to be identically zero on the algebraic part of the cohomology, γ_D is the zero map altogether, which completes the proof of a).

b) In view of (13), the result of a) shows that the $\mathbb{Q}$-cycle D on $(X\times X)_{\overline{k}}$ is homologically equivalent to zero. This property holds for l-adic cohomology, too, which implies the assertion, due to formulae (15) and (16). Note here that clearly D is numerically equivalent to zero. As D is a cycle of codimension 2, this is known to imply homological equivalence to zero with respect to any Weil cohomology theory [Lie, Corollary 1]. □

Theorem 7.7 (The splitting field and the jump character)**.** *Let X be a $K3$ surface over a number field k. Assume that the endomorphism field E of X is a CM field.*
a) *Then, for every prime number l, there is a natural isomorphism*

$$G_{X,\overline{l}}/G^0_{X,\overline{l}} \stackrel{\cong}{\longrightarrow} \mathrm{Gal}(kE/k)\,.$$

I.e., the splitting field of $G_{X,\overline{l}}/G^0_{X,\overline{l}}$ is exactly kE.
b) *The jump character $\tau_X\colon \mathrm{Gal}(\overline{k}/k)\to\{\pm1\}$ is*

$$\sigma\mapsto\begin{cases} 1\,, & \textit{if } \frac{22-\mathrm{rk}\,\mathrm{Pic}\,X_{\overline{k}}}{[E:\mathbb{Q}]} \textit{ is even}\,,\\ \begin{array}{r}\textit{sign of the natural permutation}\\ \textit{action of } \sigma\in\mathrm{Gal}(\overline{k}/k) \textit{ on the}\\ \textit{conjugates } u_1,\overline{u}_1,\ldots,u_d,\overline{u}_d \textit{ of}\\ \textit{a primitive element } u_0\in E\,,\end{array} & \textit{if } \frac{22-\mathrm{rk}\,\mathrm{Pic}\,X_{\overline{k}}}{[E:\mathbb{Q}]} \textit{ is odd}\,.\end{cases}$$

Proof. It is known that the Hodge conjecture holds for $(X\times X)(\mathbb{C})$ [RM, Theorem 5.4]. Thus, in view of Lemma 7.6.b), both assertions follow directly from the fundamental Lemma 5.6. □

Corollary 7.8. *Let X be a $K3$ surface over a number field k. Suppose that the endomorphism field of X is an imaginary quadratic field $E=\mathbb{Q}(\sqrt{-\delta})$, for $\delta\in\mathbb{N}$. Then the jump character $\tau_X\colon \mathrm{Gal}(\overline{k}/k)\to\{\pm1\}$ is given by*

$$\mathrm{Frob}_{\mathfrak{p}}\mapsto\begin{cases} 1, & \textit{if } \mathrm{rk}\,\mathrm{Pic}\,X_{\overline{k}}\equiv 2 \pmod 4\,,\\ (\frac{-\delta}{\mathfrak{p}})\,, & \textit{if } \mathrm{rk}\,\mathrm{Pic}\,X_{\overline{k}}\equiv 0 \pmod 4\,.\end{cases}$$

8. Examples

Complex multiplication.

Example 8.1. Let X_1' be the double cover of $\mathbf{P}^2_{\mathbb{Q}}$, given by

$$w^2=xyz(x+y+z)(x+2y+3z)(5x+8y+20z)$$

and X_1 the $K3$ surface obtained as the minimal desingularisation of X_1'. Then, as shown in [EJ22b, Example 5.7], the geometric Picard rank of X_1 is 16 and the endomorphism field of X_1 is $E=\mathbb{Q}(i)$. Thus, Theorem 7.7.a) implies that $G_{X_1,\overline{l}}/G^0_{X_1,\overline{l}}\cong\mathrm{Gal}(E/\mathbb{Q})=\mathbb{Z}/2\mathbb{Z}$ and Corollary 7.8 shows that the jump character τ_{X_1} is given by $(\frac{-1}{\cdot})$. Both facts have been obtained before, using other methods.

Example 8.2. Let $X_2:=\mathrm{Kum}(\mathrm{J}(C))$ be the Kummer surface associated with the Jacobian of the genus 2 curve C over $\mathbb{Q}$, given by $w^2=x^5-1$.

a) Then the geometric Picard rank of X_2 is 18.

b) The endomorphism field of X_2 is $E = \mathbb{Q}(\zeta_5)$ and the jump character τ_{X_2} is given by $(\frac{5}{\cdot})$.

Proof. a) According to [vW, §7], the Jacobian $\mathrm{J}(C)$ of C has CM by the quartic field $\mathbb{Q}(\zeta_5)$. This yields that $\mathrm{rk}\,\mathrm{NS}\,\mathrm{J}(C)_{\mathbb{C}} = [\mathbb{Q}(\zeta_5):\mathbb{Q}]/2 = 2$ [Mu, Section 21, Application III]. Finally, it is well-known (cf. [Lim] or [EJ12, Fact 4.1]) that

$$\mathrm{rk}\,\mathrm{Pic}\,X_{2,\mathbb{C}} = \mathrm{rk}\,\mathrm{Pic}\,\mathrm{Kum}(\mathrm{J}(C))_{\mathbb{C}} = \mathrm{rk}\,\mathrm{NS}\,\mathrm{J}(C)_{\mathbb{C}} + 16\,.$$

b) For the transcendental part $T \subset H^2(X_2(\mathbb{C}),\mathbb{Q})$ of the cohomology, one has an inclusion $T \hookrightarrow H^2(\mathrm{J}(C)(\mathbb{C}),\mathbb{Q}) = \bigwedge^2 H^1(C(\mathbb{C}),\mathbb{Q})$. On the right hand side, let ζ_5 act as $v \wedge v' \mapsto {}^{[\zeta_5]}v \wedge {}^{[\zeta_5]}v'$. This is well-defined and a morphism of Hodge structures. Moreover, on T, the minimal polynomial of this morphism is $(T^5-1)/(T-1)$, so there is an extension to an action of the whole of $\mathbb{Q}(\zeta_5)$. I.e., one has $E \supseteq \mathbb{Q}(\zeta_5)$ and the inclusion the other way round follows from the fact that T carries the structure of an E-vector space, which implies $[E:\mathbb{Q}] \mid (22 - \mathrm{rk}\,\mathrm{Pic}\,X_{2,\overline{\mathbb{Q}}})$. The final claim follows from Theorem 7.7.b). It may as well be obtained using [CEJ, Algorithm 2.6.1]. □

In view of these results, Proposition 5.5.a) shows that $G^0_{X_2,\bar{l}} \cong [\overline{\mathbb{Q}}_l^*]^2$. Moreover, $G_{X_2,\bar{l}}/G^0_{X_2,\bar{l}} \cong \mathrm{Gal}(E/\mathbb{Q}) = \mathbb{Z}/4\mathbb{Z}$, by Theorem 7.7.a). Based on point counting on C, which is faster than counting points on X_2, we determined $\mathrm{Tr}(\varrho_{X_2,\bar{l}}(\mathrm{Frob}_p))$, for all prime numbers $p \leq 5\cdot 10^8$. If $p \not\equiv 1 \pmod 5$ then $\mathrm{Tr}(\varrho_{X_2,\bar{l}}(\mathrm{Frob}_p)) = 0$. The experimental distribution of the traces is shown in the histogram below, plotted against the theoretical distribution, according to the Sato–Tate conjecture, the density of which is given by $\mathrm{K}(1-t^2/16)/8\pi^2$, for K the complete elliptic integral of the first kind. Cf. [EJ22b, Section 2 and formula (6)]. The histogram ignores about the primes $p \not\equiv 1 \pmod 5$, which would add a spike of mass 3/4 above 0.

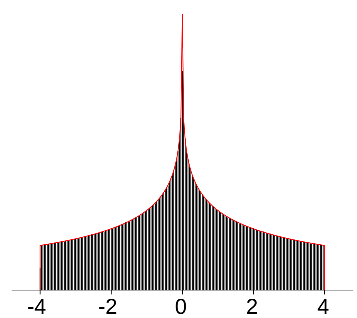

FIGURE 1. Theoretical and experimental trace distributions for the $K3$ surface X_2 over $k = \mathbb{Q}$ of geometric Picard rank 18 having CM by $E = \mathbb{Q}(\zeta_5)$

Example 8.3. Let X_3 be the Kummer surface associated with the split abelian surface $E_1 \times E_2$, for E_1 and E_2 the elliptic curves over $\mathbb{Q}$, given by $E_1\colon y^2 = x^3 + x$ and $E_2\colon y^2 = x^3 + 4x^2 + 2x$.

a) Then the geometric Picard rank of X_3 is 18.

b) The endomorphism field of X_3 is $E = \mathbb{Q}(\sqrt{2}, i)$ and the jump character τ_{X_3} is trivial.

Proof. a) One has $j(E_1) = 12^3$ and $j(E_2) = 20^3$, hence the curves have CM by $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-2})$, respectively [Co, §12.A]. This yields that $\mathrm{End}((E_1 \times E_2)_\mathbb{C}) = \mathbb{Q}(\sqrt{2}, i)$ and that $\mathrm{rk}\,\mathrm{NS}((E_1 \times E_2)_\mathbb{C}) = [\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}]/2 = 2$ [Mu, Section 21, Application III], which implies the claim.

b) For $T \subset H^2(X_3(\mathbb{C}), \mathbb{Q})$, the Künneth formula yields a natural isomorphism $T \cong H^1(E_1(\mathbb{C}), \mathbb{Q}) \otimes_\mathbb{Q} H^1(E_2(\mathbb{C}), \mathbb{Q})$. Moreover, the action of $\mathbb{Q}(\sqrt{2}, i)$ on T is given by ${}^{[i]}(v \otimes v') := {}^{[i]}v \otimes v'$ and ${}^{[\sqrt{-2}]}(v \otimes v') := v \otimes {}^{[\sqrt{-2}]}v'$. Again, the equality $E = \mathbb{Q}(\sqrt{2}, i)$ is enforced by the fact that T carries the structure of an E-vector space. The final claim is obtained using [CEJ, Algorithm 2.6.1]. □

Note that the result on the jump character as well follows from Theorem 7.7.b). Based on point counting on E_1 and E_2, we determined $\mathrm{Tr}(\varrho_{X_3,\bar{l}}(\mathrm{Frob}_p))$, for all prime numbers $p \leq 5 \cdot 10^8$. If $p \not\equiv 1 \pmod 8$ then $\mathrm{Tr}(\varrho_{X_3,\bar{l}}(\mathrm{Frob}_p)) = 0$. The experimental distribution of the traces is shown in the histogram above, plotted against the theoretical distribution, according to the Sato–Tate conjecture, which is the same as in Example 8.2. Note here that one has $G^0_{X_3,\bar{l}} \cong [\overline{\mathbb{Q}}_l^*]^2$, again. Furthermore, Theorem 7.7.a) shows that $G_{X_3,\bar{l}}/G^0_{X_3,\bar{l}} \cong \mathrm{Gal}(E/\mathbb{Q}) = [\mathbb{Z}/2\mathbb{Z}]^2$. Again, the spike above 0 of mass 3/4 is ignored.

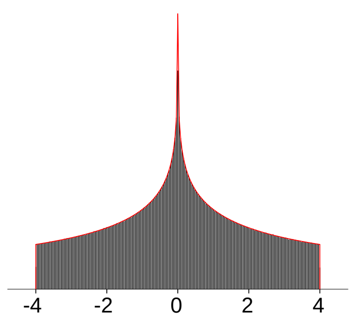

FIGURE 2. Theoretical and experimental trace distributions for the $K3$ surface X_3 over $k = \mathbb{Q}$ of geometric Picard rank 18 having CM by $E = \mathbb{Q}(\sqrt{2}, i)$

Example 8.4. Let X_4' be the double cover of $\mathbf{P}^2_\mathbb{Q}$, given by

$$w^2 = xyz(x^3 - 3x^2z - 3xy^2 - 3xyz + y^3 + 9y^2z + 6yz^2 + z^3)$$

and X_4 the $K3$ surface obtained as the minimal desingularisation of X_4'. Then, as shown in [EJ22b, Example 5.11], the geometric Picard rank of X_4 is 16. Moreover, there is strong evidence that the endomorphism field is $E = \mathbb{Q}(\zeta_9 + \zeta_9^{-1}, \sqrt{-1})$. Assuming this, Theorem 7.7.a) implies that

$$G_{X_4,\bar{l}}/G^0_{X_4,\bar{l}} \cong \mathrm{Gal}(E/\mathbb{Q}) = \mathbb{Z}/6\mathbb{Z}$$

and Theorem 7.7.b) shows that the jump character τ_{X_4} is given by $(\frac{-1}{.})$. The latter fact has been obtained before. For the former, strong evidence has been reported.

Example 8.5. Put $k := \mathbb{Q}[t]/(t^3 - t^2 - 4t + 1)$ and let X'_5 be the double cover of $\mathbf{P}^2_k$, given by

$$w^2 = xyz(x + y + z)(x + \alpha y + \beta z)(x + \gamma y + \delta z)\,,$$

for $\alpha := \overline{\frac{-26t^2-23t+16}{9}}$, $\beta := \overline{\frac{-61t^2+125t+95}{121}}$, $\gamma := \overline{\frac{-t^2-4t+11}{9}}$, and $\delta := \overline{\frac{-46t^2+5t+149}{121}}$. Let X_5 be the $K3$ surface obtained as the minimal desingularisation of X'_5. Then the geometric Picard rank of X_5 is 16.

Proof. An upper bound of 16 is provided by the reduction $X_{5,\mathfrak{p}}$ at the unique prime ideal $\mathfrak{p}$ of k of ideal norm 29, which is of geometric Picard rank 16. On the other hand, the ramification locus has 15 singular points. Thus, X_5 contains 15 (-2)-curves, which are mutually skew, and hence provide a lower bound of 16, together with the inverse image of a general line on $\mathbf{P}^2_k$. □

There is strong evidence that the endomorphism field of X_5 is $E = k(i)$. Note that the cubic field k is totally real, so E is indeed a CM field. Moreover, E is not normal over $\mathbb{Q}$, but, in agreement with Theorem 7.5, one has that $kE = k(i)$ is normal over k. If this is true then Proposition 5.5.a) shows that $G^0_{X_5,\bar{l}} \cong [\overline{\mathbb{Q}}_l^*]^3$. Furthermore, $G_{X_5,\bar{l}}/G^0_{X_5,\bar{l}} \cong \mathrm{Gal}(kE/k) = \mathrm{Gal}(k(i)/k) \cong \mathbb{Z}/2\mathbb{Z}$ and the jump character τ_{X_5} is given by $(\frac{-1}{.})$.

The evidence is as follows. We calculated the characteristic polynomial of $\mathrm{Frob}_\mathfrak{p}$ on $T_{\bar{l}} \subset H^2_{\text{ét}}(X_{5,\bar{k}}, \overline{\mathbb{Q}}_l(1))$, for all prime ideals $\mathfrak{p}$ of k of ideal norm $< 10\,000$, at which X_5 has good reduction. It turned out that, for each $\mathfrak{p}$, $\mathrm{rk}\,\mathrm{Pic}\,X_{5,\bar{\mathfrak{p}}}$ is either 16 or 22. More precisely, if $\mathfrak{p}$ is inert in $k(i)$ then one always has that $\mathrm{Tr}(\varrho_{X_5,\bar{l}}(\mathrm{Frob}_\mathfrak{p})) = 0$ and that the reduction is of geometric Picard rank 22. On the other hand, if $\mathfrak{p}$ splits in $k(i)$ then the geometric Picard rank of the reduction happens to be 16, each time. Moreover, the characteristic polynomial of $\mathrm{Frob}_\mathfrak{p}$ on $T_{\bar{l}}$ splits off two linear factors over $k(i)$. Over $k^{(n)}(i)$, it splits into linear factors completely. If $\mathfrak{p}$ is a prime of degree 3 then characteristic polynomial of $\mathrm{Frob}_\mathfrak{p}$ on $T_{\bar{l}}$ is a perfect cube of a quadratic polynomial. Furthermore, the height [AM] of the reduction $X_{5,\mathfrak{p}}$ coincides with the degree of $\mathfrak{p}$. In particular, $X_{5,\mathfrak{p}}$ is ordinary if and only if $\mathfrak{p}$ is a degree one prime ideal that splits in $k(i)$.

In addition, we calculated the trace of $\mathrm{Frob}_\mathfrak{p}$, for all primes $\mathfrak{p}$ up to ideal norm 10^7. The observation that $\mathrm{Tr}(\varrho_{X_5,\bar{l}}(\mathrm{Frob}_\mathfrak{p})) = 0$, for every inert prime $\mathfrak{p}$, extends up to this bound. Moreover, for each split prime, one has exactly

$$\nu_p(\mathrm{Tr}(\varrho_{X_5,\bar{l}}(\mathrm{Frob}_\mathfrak{p}))) = [\mathscr{O}_k/\mathfrak{p} : \mathbb{F}_p] - 1\,,$$

for $\nu_p\colon \mathbb{Q}^* \to \mathbb{Z}$ the normalised p-adic valuation and p the prime number lying below $\mathfrak{p}$. The information is summarised in the table below. Note, in particular, that the observation concerning ordinarity made above extends up to the bound of 10^7.

Behaviour of $\mathfrak{p}$ in $k(i)/k$ / Degree of $\mathfrak{p}$	inert	split
1	$\mathrm{Tr}(\varrho_{X_5,\bar{l}}(\mathrm{Frob}_\mathfrak{p})) = 0$	$\nu_p(\mathrm{Tr}(\varrho_{X_5,\bar{l}}(\mathrm{Frob}_\mathfrak{p}))) = 0$
2	not possible	$\nu_p(\mathrm{Tr}(\varrho_{X_5,\bar{l}}(\mathrm{Frob}_\mathfrak{p}))) = 1$
3	$\mathrm{Tr}(\varrho_{X_5,\bar{l}}(\mathrm{Frob}_\mathfrak{p})) = 0$	$\nu_p(\mathrm{Tr}(\varrho_{X_5,\bar{l}}(\mathrm{Frob}_\mathfrak{p}))) = 2$

TABLE 1. The traces of $\mathrm{Frob}_\mathfrak{p}$ for the $K3$ surface X_5 over a non-normal cubic number field k of geometric Picard rank 16 having CM by $E = k(i)$

Remark 8.6. Having chosen a suitable basis of $H^2(X_5(\mathbb{C}), \mathbb{Z})$, as described in [EJ22a, Corollary 3.14], the restricted period point [EJ22a, Definition 3.15] of X_5 is

$$(1 : \overline{t^2 - t - 2} : \overline{1 - t} : i : \overline{t^2 - t - 2} \cdot i : \overline{1 - t} \cdot i)\,, \tag{18}$$

up to an error of less than 10^{-229}. Here, k is considered as a subfield of $\mathbb{R}$, via the embedding, given by $\bar{t} \mapsto 0.23912\ldots$ For the $\mathbb{Q}$-linear action of $k(i)$, given by

$$\bar{t} \mapsto \begin{pmatrix} 1 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & -2 & 0 \end{pmatrix} \qquad \text{and} \qquad i \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}\,,$$

the one-dimensional vector space underlying (18) is an eigenspace, which would prove CM by $k(i)$ if we knew the period point exactly [EJ22a, Theorem 6.29.b)]. The search for this example has actually been based on period integration [EJ22a, Algorithm 5.2].

The experimental distribution of the traces for all prime ideal $\mathfrak{p}$ in k of ideal norm $< 10^7$ is shown in the histogram below, plotted against the theoretical distribution, according to the Sato–Tate conjecture. The spike of mass 1/2 above 0 is not shown. The density of the theoretical distribution agrees, up to scaling, with the one discussed in [EJ22b, Section 3, second of Examples B]. There is an explicit formula, which is, however, rather complicated [EJ22b, formula (7)].

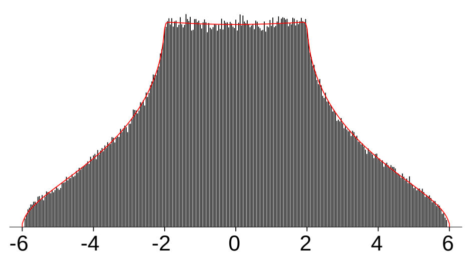

FIGURE 3. Theoretical and experimental trace distributions for the $K3$ surface X_5 over a non-normal cubic number field k of geometric Picard rank 16 having CM by $E = k(i)$

Real multiplication.

In the generic and RM cases, it may easily happen that the splitting field of $G_{X,\bar{l}}/G^0_{X,\bar{l}}$ strictly contains kE. Cf. [EJ22b, Examples 5.4, 5.8 and 5.10]. In the example below, however, equality holds, at least conjecturally.

Example 8.7. Let X'_6 be the double cover of $\mathbf{P}^2_{\mathbb{Q}}$, given by

$$w^2 = -1974xyz(x^3 - 14x^2z + 11xy^2 - xz^2 + 12y^3 - 14y^2z - 12yz^2 + 14z^3)$$

and X_6 the $K3$ surface obtained as the minimal desingularisation of X'_6.

a) Then the geometric Picard rank of X_6 is 16.

b) The endomorphism field of X_6 is at most quadratic.

Proof. The surface X_6 is the quadratic twist of the surface $\widetilde{X}_6$, presented in [EJ22b, Example 5.10], by the twist factor $(-1974) = -2{\cdot}3{\cdot}7{\cdot}47$. I.e., the two surfaces are geometrically isomorphic. In fact, they are isomorphic over $K = \mathbb{Q}(\sqrt{-1974})$. In particular, the geometric Picard ranks as well as the endomorphism fields are the same. □

There is strong evidence that the endomorphism field of X_6 is $E = \mathbb{Q}(\sqrt{3})$. Note that $\widetilde{X}_6 = V^{(3)}_{1,2}$ in the notation of [EJ16, Conjectures 5.2] and $\widetilde{X}_6 = X_6$ in the notation of [EJ22b]. Thus, conjecturally, $G^0_{X_6,\bar{l}} \cong [\mathrm{SO}_3(\overline{\mathbb{Q}}_l)]^2$, for any prime l.

Concerning the component group, we claim that $G_{X_6,\bar{l}}/G^0_{X_6,\bar{l}}$ is of order 2. For this, recall the inclusion

$$G_{X_6,\bar{l}}/G^0_{X_6,\bar{l}} \hookrightarrow \mathrm{N}_{\mathrm{O}_6(\overline{\mathbb{Q}}_l)}([\mathrm{SO}_3(\overline{\mathbb{Q}}_l)]^2)/[\mathrm{SO}_3(\overline{\mathbb{Q}}_l)]^2 \cong (\mathbb{Z}/2\mathbb{Z})^2 \rtimes S_2$$

into the dihedral group of order 8. Moreover, one clearly has

$$\#X'_6(\mathbb{F}_p) = \begin{cases} \#\widetilde{X}'_6(\mathbb{F}_p)\,, & \text{if } (\frac{-1974}{p}) = 1\,, \\ 2(p^2+p+1) - \#\widetilde{X}'_6(\mathbb{F}_p)\,, & \text{if } (\frac{-1974}{p}) = -1 \end{cases}$$

and, consequently,

$$\varrho_{X_6,\bar{l}} = \varrho_{\widetilde{X}_6,\bar{l}} \cdot (\tfrac{-1974}{\cdot})\,. \qquad (19)$$

Thus, the experimental observations described in [EJ22b, Section 5] carry over as follows.

One has $\mathrm{Tr}(\varrho_{X_6,\bar{l}}(\mathrm{Frob}_p)) = 0$ for all primes $p = \pm 5 \pmod{12}$, at least as long as $p < 5{\cdot}10^8$. These are exactly the primes such that the jump character τ_{X_6} evaluates to (-1) at the corresponding Frobenii, hence the component group is bound to elements of the types $\left(\begin{smallmatrix} + & 0 \\ 0 & + \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} - & 0 \\ 0 & - \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & + \\ + & 0 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 0 & - \\ - & 0 \end{smallmatrix}\right)$. Furthermore, formula (19) shows that, in view of the corresponding observation for $\widetilde{X}_6$ [EJ22b, comments below Example 5.10],

$$\varrho_{X_6,\bar{l}}(\mathrm{Frob}_p) \in [\mathrm{O}_3^-(\overline{\mathbb{Q}}_l)]^2 \iff p \equiv \pm 1 \pmod{12} \text{ and } (\tfrac{3}{p}) = -1\,,$$

which is contradictory. Hence, the component group $G_{X_6,\bar{l}}/G^0_{X_6,\bar{l}}$ is bound to order 2.

The experimental distribution of the traces for all primes $p < 5 \cdot 10^8$ is shown in the histogram below, plotted against the theoretical distribution, according to the Sato–Tate conjecture, the density of which is given by

$$\tfrac{1}{8\pi^2}\big((2-t)\mathrm{K}(1-\tfrac{(t-2)^2}{16}) + 4\mathrm{E}(1-\tfrac{(t-2)^2}{16})\big)\,,$$

for K and E the complete elliptic integrals of the first and second kinds. Cf. [EJ22b, Section 3]. The spike of mass 1/2 above 0 is not shown.

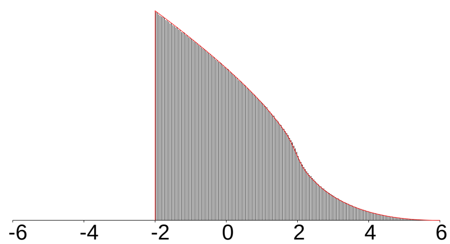

FIGURE 4. Theoretical and experimental trace distributions for the $K3$ surface X_6 over $k = \mathbb{Q}$ of geometric Picard rank 16 having RM by $E = \mathbb{Q}(\sqrt{3})$

Remark 8.8. Put $X_{6'} := (X_6)_k$, for $k = \mathbb{Q}(\sqrt{3})$. Then the geometric Picard rank and the endomorphism field E are the same as for X_6. However, if $E = \mathbb{Q}(\sqrt{3})$ is indeed true then $G_{X_{6'},\bar{l}} = [\mathrm{SO}_3(\overline{\mathbb{Q}}_l)]^2$ is connected. I.e., in the histogram above, the spike disappears. As shown in Theorem 4.5, this happens if and only if $k \supseteq E$.

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