ON THE CHARACTERISTIC POLYNOMIAL OF THE FROBENIUS ON ÉTALE COHOMOLOGY

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Abstract. Let X be a smooth proper variety of even dimension d over a finite field. We establish a restriction on the value at (-1) of the characteristic polynomial of the Frobenius on the middle-dimensional étale cohomology of X with coefficients in $\mathbb{Q}_l(d/2)$.

1. Introduction

Let X be a smooth proper variety over a finite field $\mathbb{F}_q$ of characteristic $p > 0$. Then the geometric Frobenius Frob operates linearly on the l-adic cohomology vector spaces $H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(j))$. The characteristic polynomial of Frob has rational coefficients [11, Théorème (1.6)].

By far not every polynomial $\Phi \in \mathbb{Q}[T]$ may occur as the characteristic polynomial of the Frobenius on a smooth proper variety. The following conditions were established essentially in the 70s of the last century, in the course of working out A. Grothendieck's ideas in Algebraic Geometry.

Theorem 1.1 (Deligne, Mazur, Ogus)**.** *Let X be a smooth proper variety over a finite field $\mathbb{F}_q$ of characteristic p. For $i, j \in \mathbb{Z}$, denote by*

$$\Phi^{(i)}_j = T^N + a^{(i)}_1 T^{N-1} + \cdots + a^{(i)}_{N-1} T + a^{(i)}_N$$

the characteristic polynomial of Frob *on $H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(j))$. Then, for every r, one has $a^{(i)}_r \in \mathbb{Q}$ and this value is independent of $l \neq p$. Moreover,*

a) *every complex zero of $\Phi^{(i)}_j$ is of absolute value $q^{i/2-j}$.*

b) *If i is odd then all real zeroes of $\Phi^{(i)}_j$ are of even multiplicity.*

c) *For every $l \neq p$, the zeroes of $\Phi^{(i)}_j$ are l-adic units, i.e. units in a suitable extension of $\mathbb{Z}_l$.*

d) *Put $h_{i-m,m} := \dim H^{i-m}(X, \Omega^m_X)$ and define the step-function $G^{(i)}\colon [0, N] \to \mathbb{R}$ by*

$$G^{(i)}(t) = \begin{cases} 0 & \text{for } t \leq h_{i,0}\,, \\ n & \text{for } h_{i,0} + \cdots + h_{i-n+1,n-1} < t \leq h_{i,0} + \cdots + h_{i-n,n}\,. \end{cases}$$

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Then, for $r = 1, \dots, N$, one has $a_r^{(i)} = 0$ or

$$\nu_q(a_r^{(i)}) \geq \int_0^r [G^{(i)}(t) - j]\, dt\,.$$

Here, ν_q is the non-archimedean valuation such that $\nu_q(q) = 1$.

Remarks 1.2. i) We will provide references and proof sketches at the beginning of Section 2.

ii) Assertion a) immediately implies that $\Phi_j^{(i)} \in \mathbb{Q}[T]$ fulfills the functional equation

$$T^N \Phi(q^{i-2j}/T) = \pm q^{\frac{N}{2}(i-2j)} \Phi(T)$$

for $N := \dim H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(j))$. Indeed, on both sides, there are polynomials with leading term $\pm q^{\frac{N}{2}(i-2j)} T^N$. They have the same zeroes as, with z, the number $\overline{z} = \frac{q^{i-2j}}{z}$ is a zero of Φ, too.

Furthermore, by b), the plus sign always holds when i is odd. The statements c) and d) together show $\Phi_j^{(i)} \in \mathbb{Z}[T]$ when $j \leq 0$.

iii) For X projective and i even, we also have

$$\Phi_j^{(i)}(q^{i/2-j}) = 0\,. \tag{1}$$

Indeed, there is the $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-invariant cycle given by the intersection of $i/2$ hyperplanes. The cycle map [10, Cycle, Théorème 2.3.8.iii)] yields a non-trivial Galois invariant element of $H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(i/2))$.

Remark 1.3. Consider the case that $i = 1$. Then $N = \dim H^1_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$ is always even [12, Corollaire (4.1.5)]. On the other hand, let a polynomial $\Phi \in \mathbb{Z}[T]$ be given that is of even degree and fulfills assertions a), b), and c). Then, by the main theorem of T. Honda [23], there exists an abelian variety A such that the eigenvalues of Frob on $H^1_{\text{ét}}(A_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$ are exactly the zeroes of Φ. One may enforce that the characteristic polynomial is a power of Φ, and, typically, Φ itself may be realized.

1.4. We will show in this note that the same is not true in general for $i > 1$. In fact, for the characteristic polynomial of Frob on the middle cohomology of a variety of even dimension, we will establish a further condition, which is arithmetic in nature and independent of Theorem 1.1, as well as of formula (1).

Theorem 1.5. *Let X be a smooth proper variety of even dimension d over a finite field $\mathbb{F}_q$ of characteristic p and $\Phi = \Phi_{d/2}^{(d)} \in \mathbb{Q}[T]$ be the characteristic polynomial of* Frob *on $H^d_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(d/2))$. Put $N := \deg \Phi$.*

Then $(-2)^N \Phi(-1)$ is a square or p times a square in $\mathbb{Q}$.

Remark 1.6. For X a surface, this result may be deduced from the Tate conjecture, via the Artin-Tate formula. Cf. Proposition 3.11, below.

1.7. The correct exponent of p, may, at least for $p \neq 2$, be described as follows.

Definition. We put
$$a(X) := \sum_{m=0}^{\frac{d}{2}-1} (\tfrac{d}{2} - m) h'_{d-m,m}\,,$$
for $(h'_{d-m,m})_m$ the abstract Hodge numbers of X in degree d [30, Section 4].

Remarks 1.8. a) Recall that the abstract Hodge numbers are defined as follows. The crystalline cohomology groups $H^i(X/W)$ are finitely generated W-modules, for $W := W(\mathbb{F}_q)$ the Witt ring. They are acted upon by the absolute Frobenius $\mathbf{F}$, the corresponding map is only $\mathbf{F}$-semilinear [8, Exposé I, 2.3.5].

Put $H := H^i(X/W)/\text{tors}$. Then, as $\mathbf{F}\colon H \to H$ is injective, $H/\mathbf{F}H$ is a W-module of finite length. By the classical invariant factor theorem, there is a unique sequence of integers such that $H/\mathbf{F}H \cong \bigoplus\limits_{m>0} (W/p^m W)^{h'_{i-m,m}}$. Finally, one defines $h'_{i,0} := \text{rk}_W H - \sum\limits_{m>0} h'_{i-m,m}$.

Observe that $a(X)$ is a geometric quantity. It depends only on the base extension $X_{\overline{\mathbb{F}}_q}$.

b) Suppose that X is such that all $H^i(X/W)$ are torsion-free and that the conjugate spectral sequence $E_2^{jm} := H^j(X, \mathscr{H}^m(\Omega^\bullet_{X/\mathbb{F}_q})) \Longrightarrow H^i_{\text{dR}}(X)$ degenerates at E_2. Then $h'_{i-m,m} = h_{i-m,m}\,(= \dim H^{i-m}(X, \Omega^m_X))$ [5, Lemma 8.32].

For complete intersections, both assumptions hold ([15, Exposé XI, Théorème 1.5] together with [5, Lemma 8.27.2]). Moreover, the second assumption is automatically fulfilled when $\dim X \leq p$ and X lifts to W ([14, Corollaire 2.4] and [5, Lemma 8.27.2]).

c) Suppose that X is of Hodge-Witt type in degree d, i.e., that the Serre cohomology groups $H^j(X, W\Omega^m_X)$ [36] are finitely generated W-modules for $j + m = d$. The assertion of Theorem 1.9 below may then be formulated entirely in terms of the characteristic polynomial Φ. In fact, denote the zeroes of Φ by $z_1, \ldots, z_N$. Then
$$a(X) = -\sum_{\nu_q(z_i)<0} \nu_q(z_i)$$
[Corollary 2.16]. This case includes all varieties that are ordinary in degree d [25, Définition IV.4.12].

Theorem 1.9. *Let X be a smooth proper variety of even dimension d over a finite field $\mathbb{F}_q$ of characteristic $p \neq 2$ and $\Phi = \Phi^{(d)}_{d/2} \in \mathbb{Q}[T]$ be the characteristic polynomial of* Frob *on $H^d_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(d/2))$. Put $N := \deg \Phi$.*
Then $(-2)^N q^{a(X)} \Phi(-1)$ is a square in $\mathbb{Q}$.

Remarks 1.10. i) It seems not unlikely that an analogous result is true in characteristic 2, too. The difficulties that arise may well be of purely technical nature. Cf. Remark 2.14, below.

ii) The assertions may easily be formulated for an arbitrary Tate twist. But this does not lead to anything new. Also, the twist by $d/2$ appears to be natural as the operation of Frob on $H^d_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(d/2))$ is orthogonal with respect to a $\mathbb{Q}_l$-valued symmetric, bilinear pairing.

Remarks 1.11. i) One reason for our interest in these conditions is that they simplify the actual computation of the characteristic polynomial Φ for a given variety X. Such computations usually involve point counting on X over several extensions of the base field, cf. Examples 3.13 and 3.15. The conditions given redundantise the most expensive counting steps.

ii) Applications of the characteristic polynomials Φ to varieties in characteristic zero include the determination of the Néron-Severi rank. The interested reader might consult the articles [28], [19], and [20].

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2. The proofs

2.1. Proof of Theorem 1.1. a) This was first proven by P. Deligne in [11, Théorème (1.6)] for the projective case and later in [12, Corollaire (3.3.9)], in general. The assertion was formulated by A. Weil as a part of his famous conjectures.

b) If X is projective then, by the Hard Lefschetz theorem [12, Théorème (4.1.1)] and Poincaré duality, there is a non-degenerate pairing

$$H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(j)) \times H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(j)) \to \mathbb{Q}_l(2j-i)$$

that is compatible with the operation of Frob. It is alternating as i is odd. The assertion follows directly from this. Cf. the remarks after [12, Corollaire (4.1.5)]. The proper non-projective case has only recently been settled by J. Suh [40, Corollary 2.2.3].

c) As Frob operates on $H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_l(j))$, the l-adic valuations of the eigenvalues are clearly non-negative. Poincaré duality implies the assertion, cf. [11, (2.4)].

d) This statement was originally known as Katz's conjecture. The usual formulation is that the Newton polygon of $\Phi_0^{(i)}$ lies above the Hodge polygon of weight i. Proofs are due to B. Mazur [31] and A. Ogus [5, Theorem 8.39]. □

Notation 2.2. For R an integral domain, K its field of fractions, and H an R-module, we will write $H_K := H \otimes_R K$.

Lemma 2.3. *Let R be a principal ideal domain, K its field of fractions, and H a free R-module of finite rank, equipped with a perfect, R-bilinear, symmetric pairing $H \times H \to R$. Denote its K-bilinear extension by $\langle .\,,.\rangle \colon H_K \times H_K \to K$.*

Furthermore, let a K-linear map $\sigma\colon H_K \to H_K$ be given that is orthogonal with respect to the pairing, i.e., $\langle \sigma(x), \sigma(y)\rangle = \langle x, y\rangle$ for every $x, y \in H_K$.

Put

$$B_0(H) := [H/[H \cap (1-\sigma)H]]_{\rm tors}\,.$$

a) *Then there is a non-degenerate, skew-symmetric R-bilinear pairing*

$$(\,.\,,.\,)\colon B_0(H) \times B_0(H) \to K/R\,.$$

b) *Suppose* $\operatorname{char} K \neq 2$ *and* $\langle x,x\rangle \in 2R$ *for every* $x \in H \cap (1-\sigma)H_K$. *Then* $(\,.\,,.\,)$ *is alternating. In particular, the length of* $B_0(H)$ *is even.*

Remarks 2.4. i) Observe that $x \in H$ represents an element of $B_0(H)$ if and only if $x \in H \cap (1-\sigma)H_K$.

ii) For $x, y \in H_K$ arbitrary, one has $\langle (1-\sigma)x, \sigma y\rangle = -\langle x, (1-\sigma)y\rangle$. In particular, as $\sigma\colon H_K \to H_K$ is bijective, $x \in \ker(1-\sigma)$ if and only if $x \in ((1-\sigma)H_K)^\perp$. I.e., $(1-\sigma)H_K$ is exactly the set of all elements perpendicular to the eigenspace $H_{K,1}$. This fact is rather obvious, let us nevertheless emphasize that it is true whether σ is semisimple or not.

2.5. Proof of Lemma 2.3. a) *Definition.* The pairing is defined as follows. For $a, b \in B_0(H)$, choose representatives $x, y \in H$. Let $y' \in H_K$ be such that $y = (1-\sigma)y'$. Then $(a,b) := \langle x, y'\rangle \bmod R$.

Well-definedness. For two representatives $x_1, x_2 \in H$, we have $x_1 - x_2 = (1-\sigma)v$ for some $v \in H$. Thus,

$$\begin{aligned}\langle x_1 - x_2, y'\rangle = \langle (1-\sigma)v, y'\rangle = \langle (1-\sigma)\sigma v, \sigma y'\rangle &= -\langle \sigma v, (1-\sigma)y'\rangle \\ &= -\langle v - x_1 + x_2, y\rangle \in R\,,\end{aligned}$$

as both sides are in H. On the other hand, for two representatives $y_1', y_2' \in H_K$, we have $(1-\sigma)(y_1' - y_2') = (1-\sigma)w \in H$ for a suitable $w \in H$. Therefore, $\langle x, y_1' - y_2'\rangle = \langle x, w\rangle + \langle x, k\rangle$ for some $k \in H_{K,1}$. The first summand is in R, as both sides are elements of H. The second summand vanishes, since $x \in (1-\sigma)H_K$. It is clear that $(\,.\,,.\,)$ is R-bilinear.

Non-degeneracy. For $0 \neq b \in B_0(H)$, one has a representative y and some $y' \in H_K$ such that $y = (1-\sigma)y'$. As $y \notin (1-\sigma)H$, we see $y' \notin H + H_{K,1}$. The goal is to find some $x \in H \cap (1-\sigma)H_K$ such that $\langle x, y'\rangle \notin R$.

For this, we observe that the perfect pairing induces an isomorphism

$$H_K \xrightarrow{\cong} \operatorname{Hom}(H, K)\,.$$

Under this map, $H_{K,1} \cong \operatorname{Hom}(H/[H \cap (1-\sigma)H_K], K)$. Furthermore,

$$H + H_{K,1} \cong \{\alpha \in \operatorname{Hom}(H,K) \mid \alpha(H \cap (1-\sigma)H_K) \subseteq R\}\,.$$

Indeed, as $H \cong \operatorname{Hom}(H,R)$, the inclusion "$\subseteq$" is obvious. The other inclusion follows from the fact that $H \cap (1-\sigma)H_K$ is a direct summand of H. The homomorphism $\alpha|_{H\cap(1-\sigma)H_K}\colon H \cap (1-\sigma)H_K \to R$ may thus be extended to a homomorphism $\alpha'\colon H \to R$, corresponding to an element of H. The difference $\alpha - \alpha'$ vanishes on $H \cap (1-\sigma)H_K$, hence is defined by an element of $H_{K,1}$.

Now, as $y' \notin H + H_{K,1}$, the corresponding homomorphism does not send $H \cap (1-\sigma)H_K$ to R. I.e., there is some $x \in H \cap (1-\sigma)H_K$, not mapped to R. This is exactly our claim.

Skew-symmetry. Let $a, b \in B_0(H)$ and choose representatives $x, y \in H$. There are $x', y' \in H_K$ such that $x = (1-\sigma)x'$ and $y = (1-\sigma)y'$. Then

$$\langle x, y' \rangle + \langle x', y \rangle = \langle x' - \sigma x', y' \rangle + \langle x', y' - \sigma y' \rangle = \langle x' - \sigma x', y' - \sigma y' \rangle = \langle x, y \rangle \in R\,.$$

b) For $a \in B_0(H)$ choose a representative $x \in H$ and $x' \in H_K$ such that $x = (1-\sigma)x'$. Then

$$\begin{aligned} 2(a,a) = 2\langle x, x' \rangle = [\langle x', x' \rangle - \langle \sigma x', x' \rangle] + [\langle \sigma x', \sigma x' \rangle - \langle x', \sigma x' \rangle] \\ = \langle x' - \sigma x', x' - \sigma x' \rangle = \langle x, x \rangle \in 2R\,, \end{aligned}$$

hence $(a,a) = 0$.

Evenness of $\ell(B_0(H))$. First note that $B_0(H)$ is a finitely generated torsion module over a principal ideal domain. This implies that $\ell(B_0(H))$ is finite.

It is known [37, Exercise 10.20] that, for such a module M, the existence of a non-degenerate alternating pairing implies that $\ell(M)$ is even. The argument is essentially as follows.

Assume the contrary and let M be a counterexample of minimal length. Choose $x \in M$ such that $Rx \subseteq M$ is of length one. I.e., $\mathfrak{m} := \mathrm{Ann}_R(x)$ is a maximal ideal. Then $L_x := \langle x, . \rangle \colon M \to K/R$ is a nonzero linear form, the image of which is $\mathfrak{m}^{-1}/R \cong R/\mathfrak{m}$. Clearly, one has $x \in \ker L_x$.

There is an alternating form induced on $[\ker L_x]/(x)$, which is again non-degenerate. Moreover, $\ell(\ker L_x) = \ell(M) - 1$ and $\ell([\ker L_x]/(x)) = \ell(M) - 2$. This is a contradiction. □

Remarks 2.6. i) If 2 is a unit in R then the assumption of b) is automatically fulfilled.

ii) When $R = \mathbb{Z}_l$, $\sigma(H) \subseteq H$, and σ is semisimple at 1, this result was proven by Yu. G. Zarhin in [43, 3.3 and Lemma 3.4.1]. It is implicitly contained in the work of J. W. S. Cassels [6].

iii) Most of our applications will be based on the following corollary.

Corollary 2.7. *Let (R, ν) be a normalized discrete valuation ring, k its residue field, which we assume to be finite of characteristic $\neq 2$, K the field of fractions, and H a free R-module of finite rank. Suppose there is a perfect, R-bilinear, symmetric pairing $H \times H \to R$ and denote its K-bilinear extension by $\langle . , . \rangle \colon H_K \times H_K \to K$.*

Furthermore, let a K-linear map $\sigma \colon H_K \to H_K$ be given that is orthogonal with respect to the pairing and such that 1 is not among its eigenvalues.

a) *Then $\ell(\sigma(H) + H/H) + \nu(\det(1-\sigma))$ is even.*

b) *In particular, if $\sigma(H) \subseteq H$ then $\nu(\det(1-\sigma))$ is even.*

Proof. a) We write $q := \#k$. According to the definition, the modulus [16, section 14.3] of the map $(1-\sigma)\colon H_K \to H_K$ may be computed as $q^{\ell(M/H)-\ell(M/(1-\sigma)H)}$ for every R-module $M \supseteq H, (1-\sigma)H$ that is chosen such that the two lengths are finite. Moreover, by [16, Exercise 14.3.6], that modulus is equal to $q^{-\nu(\det(1-\sigma))}$. Consequently,

$$\nu(\det(1-\sigma)) = \ell([\sigma(H)+H]/(1-\sigma)H) - \ell(\sigma(H)+H/H)\,.$$

On the other hand, all the assumptions of Lemma 2.3.b) are fulfilled. As 1 is not an eigenvalue of σ, $H/[H\cap(1-\sigma)H]$ is purely torsion. Thus, we have that $\ell(H/[H\cap(1-\sigma)H])$ is even. Furthermore,

$$H/[H\cap(1-\sigma)H] \cong [H+(1-\sigma)H]/(1-\sigma)H = [\sigma(H)+H]/(1-\sigma)H\,.$$

I.e., $\ell([\sigma(H)+H]/(1-\sigma)H)$ is even, too. The assertion follows.

b) is an immediate consequence of a). □

2.8. In order to illustrate the strength of Corollary 2.7, let us show an application to modules of rank two, the smallest non-trivial case. The fact obtained belongs to the not-so-well-known results on real quadratic number fields. Cf. [44, p. 118].

Corollary. *Let* $K = \mathbb{Q}(\sqrt{d})$ *be a quadratic number field and* $\varepsilon \in K$ *a unit of norm* $(+1)$. *Then* $\mathrm{N}(1-\varepsilon) = 2 - \mathrm{tr}(\varepsilon)$ *is a product of some primes dividing the discriminant, a perfect square, and, possibly, a factor* 2 *and a minus sign.*

Proof. As K is a quadratic number field, the norm $\mathrm{N}\colon \mathscr{O}_K \to \mathbb{Z}$ is a quadratic form. The multiplication map $\cdot\varepsilon\colon \mathscr{O}_K \to \mathscr{O}_K$ is compatible with this form and, therefore, orthogonal with respect to the symmetric, bilinear form $\langle\,.\,,.\rangle\colon \mathscr{O}_K \times \mathscr{O}_K \to \mathbb{Z}$ associated to N.

The same is true for the corresponding $\mathbb{Z}_l$-valued pairings between the l-adic completions of $\mathscr{O}_K$, for l any prime number. As these pairings are perfect as long as l does not divide the discriminant of K, the assertion follows from Corollary 2.7.b). □

2.9. Proof of Theorem 1.5. We clearly have that $(-2)^N\Phi(-1) \in \mathbb{Q}$. Furthermore, we may assume that (-1) is not among the zeroes of Φ as, otherwise, the assertion is true, trivially.

Then $(-2)^N\Phi(-1) > 0$. Indeed, as $\Phi \in \mathbb{R}[T]$ and there is no real zero different from 1,

$$(-2)^N\Phi(-1) = 2^N(1+z_1)\cdot\ldots\cdot(1+z_N)$$

is the product of several factors of the form $z\overline{z} = |z|^2$ for $z \in \mathbb{C}$ and some factors that are equal to 2. To prove the assertion, we will show that $(-2)^N\Phi(-1)$ is of even l-adic valuation for every prime number $l \neq p$.

Put $H := H^d_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_l(d/2))/\text{tors}$. By Poincaré duality [2, Exp. XVIII, formule (3.2.6.2)], cf. [38, Chap. 6, Sec. 2, Theorem 18 and Chap. 5, Sec. 5, Theorem 3], the bilinear pairing

$$\langle\,.\,,.\rangle\colon H \times H \longrightarrow H^{2d}_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_l(d)) \xrightarrow{\cong} \mathbb{Z}_l\,,$$

given by cup product and trace map, is perfect. As d is even, it is symmetric, too. The operation of Frob on $H_{\mathbb{Q}_l}$ is orthogonal with respect to the $\mathbb{Q}_l$-linear extension of this pairing.

First case. $l \neq 2$.

The operation of $(-\operatorname{Frob})$ is orthogonal with respect to the pairing, too. As 1 is not among its eigenvalues, Corollary 2.7.b) shows that $\nu_l(\det(1+\operatorname{Frob})) = \nu_l(\Phi(-1))$ is even.

Second case. $l = 2$.

Here, the argument is a bit more involved. First, note that the tangent sheaf $\mathscr{T}_X$ of X is defined over the base field $\mathbb{F}_q$. This shows that the Chern classes $c_i(\mathscr{T}_X) \in H^{2i}_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_2(i))$ are invariant under Frob. We therefore see from Lemma 2.10 that there is a Frob-invariant element $\omega \in H^d_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_2(d/2))$ such that $\langle \omega, x \rangle + \langle x, x \rangle \in 2\mathbb{Z}_2$ for every $x \in H^d_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_2(d/2))$.

For $x \in (1-\operatorname{Frob})H_{\mathbb{Q}_2}$, the fact that ω is Frob-invariant implies $\langle \omega, x \rangle = 0$. Hence, $\langle x, x \rangle \in 2\mathbb{Z}_2$ for $x \in H \cap (1-\operatorname{Frob})H_{\mathbb{Q}_2}$. According to Lemma 2.3.b), $[H/(1-\operatorname{Frob})H]_{\text{tors}}$ is of even length.

An application to $X_{\mathbb{F}_{q^2}}$ shows that $[H/(1-\operatorname{Frob}^2)H]_{\text{tors}}$ is of even length, too. Lemma 2.11 now yields the assertion. □

Lemma 2.10. *For every even $d > 0$, there exists a polynomial $P_d \in \mathbb{Z}[T_0, \ldots, T_{d/2}]$, weighted homogeneous of degree d for T_i of weight $2i$, such that the following is true.*

For a smooth projective variety X of dimension d over a finite field $\mathbb{F}_q$ of characteristic $\neq 2$, put $\omega := P_d(c_0(\mathscr{T}_X), \ldots c_{d/2}(\mathscr{T}_X)) \in H^d_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_2(d/2))$ for $\mathscr{T}_X$ the tangent sheaf and c_i the i-th Chern class. Then

$$\langle \omega, x \rangle + \langle x, x \rangle \in 2\mathbb{Z}_2$$

for every $x \in H^d_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_2(d/2))$.

Proof. This is a standard result in topology, cf. [33, §§8 and 11], and carries over to the étale site. Denote by $\varepsilon\colon H^{2*}_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_2(*)) \longrightarrow H^{2*}_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}/2\mathbb{Z})$ the reduction map. Furthermore, for simplicity, write $c_i := c_i(\mathscr{T}_X) \in H^{2i}_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}_2(i))$ for the Chern classes.

For $k \in \mathbb{N}_0$, let $\nu_{2k} \in H^{2k}_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}/2\mathbb{Z})$ be the $2k$-th Wu class of X [41, p. 578]. If X is of dimension d then, according to the very definition of the Wu class,

$$\nu_d \cup x + x \cup x = \operatorname{Sq}^d(x) + \operatorname{Sq}^d(x) = 0$$

for every $x \in H^d_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}/2\mathbb{Z})$ [41, Prop. 2.2.(2)]. We will inductively construct polynomials $P_{2k} \in \mathbb{Z}[T_0, \ldots, T_k]$ such that $\varepsilon(P_{2k}(c_0(\mathscr{T}_X), \ldots c_{d/2}(\mathscr{T}_X))) = \nu_{2k}$.

For every $k \in \mathbb{N}_0$, there is the formula of Wu [41, Proposition 0.5],

$$\operatorname{Sq}^0(\nu_{2k}) + \operatorname{Sq}^2(\nu_{2k-2}) + \cdots + \operatorname{Sq}^{2k}(\nu_0) = \varepsilon(c_k)\,.$$

Moreover, for the Steenrod squares of the Chern classes, there are the formulas

$$\mathrm{Sq}^{2j}(\varepsilon(c_i)) = \varepsilon(c_j)\varepsilon(c_i) + \binom{2j-2i}{2}\varepsilon(c_{j-1})\varepsilon(c_{i+1}) + \cdots + \binom{2j-2i}{2j}\varepsilon(c_0)\varepsilon(c_{i+j}).$$

Indeed, these follow in a purely formal manner from the definitions of Chern classes and Steenrod squares, cf. [33, Problem 8-A].

As $\mathrm{Sq}^0 = \mathrm{id}$, Wu's formula implies that we may choose $P_0(T_0) := T_0$. Furthermore, having $P_0, \ldots, P_{2k-2}$ already constructed, it shows that

$$\begin{aligned} \nu_{2k} = \varepsilon(c_k) - \sum_{i=0}^{k} \mathrm{Sq}^{2i}(\nu_{2k-2i}) &= \varepsilon(c_k) - \sum_{i=0}^{k} \mathrm{Sq}^{2i}(\varepsilon(P_{2k-2i}(c_0, \ldots, c_{k-i}))) \\ &= \varepsilon(c_k) - \sum_{i=0}^{k} P_{2k-2i}(\mathrm{Sq}^{2i}(\varepsilon(c_0), \ldots, \mathrm{Sq}^{2i}(\varepsilon(c_{k-i})))) \,. \end{aligned}$$

Plugging into this the formula for the Steenrod squares of the Chern classes, we see that ν_{2k} is the reduction of a polynomial expression in $c_0, c_1, \ldots, c_k$. □

Lemma 2.11. *Let (R, ν) be a normalized discrete valuation ring of characteristic 0, k its residue field, which we assume to be finite, K the field of fractions, and H a free R-module of finite rank, equipped with a non-degenerate, symmetric K-bilinear pairing $\langle . , . \rangle \colon H_K \times H_K \to K$.*

Moreover, let an R-linear map $\sigma \colon H \to H$ be given that is orthogonal with respect to the pairing and does not have the eigenvalue (-1). Suppose that $[H/(1-\sigma)H]_{\mathrm{tors}}$ and $[H/(1-\sigma^2)H]_{\mathrm{tors}}$ are of even lengths.

Then

$$\nu(\det(1+\sigma)) \equiv \nu(2) \cdot \mathrm{rk}\, H \pmod 2 \,.$$

Proof. We will prove this technical lemma in several steps.

First step. $(1-\sigma)H/(1-\sigma^2)H$ is of even length.

Since (-1) is not an eigenvalue of σ, the map $(1+\sigma)\colon H_K \to H_K$ is a bijection. In particular, $(1-\sigma)H_K = (1-\sigma^2)H_K$. Furthermore, there is the commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & (1-\sigma)H/(1-\sigma^2)H & \longrightarrow & H/(1-\sigma^2)H & \longrightarrow & H/(1-\sigma)H & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & H/H\cap(1-\sigma^2)H_K & = & H/H\cap(1-\sigma)H_K & \to & 0\,. \end{array}$$

As the vertical arrows are surjective, the 9-lemma yields exactness of

$$0 \longrightarrow (1-\sigma)H/(1-\sigma^2)H \longrightarrow [H/(1-\sigma^2)H]_{\mathrm{tors}} \longrightarrow [H/(1-\sigma)H]_{\mathrm{tors}} \longrightarrow 0\,.$$

Since $\ell([H/(1-\sigma)H]_{\mathrm{tors}})$ and $\ell([H/(1-\sigma^2)H]_{\mathrm{tors}})$ are even, we see that the length of $(1-\sigma)H/(1-\sigma^2)H$ is even, too. This is the claim.

Second step. $\nu(\det(1+\sigma)|_{(1-\sigma)H_K})$ is even.
Both, $(1-\sigma)H$ and $(1-\sigma^2)H$, are R-submodules of maximal rank in the K-vector space $(1-\sigma)H_K$. Moreover, $(1-\sigma^2)H = (1+\sigma)|_{(1-\sigma)H_K}[(1-\sigma)H]$. Hence, the modulus of $(1+\sigma)|_{(1-\sigma)H_K}$ is $q^{-\ell((1-\sigma)H/(1-\sigma^2)H)}$ for $q := \#k$.

On the other hand, by [16, Exercise 14.3.6], this modulus is $q^{-\nu(\det(1+\sigma)|_{(1-\sigma)H_K})}$. The result established in the first step implies the claim.

Third step. $\nu(\det(1+\sigma)) \equiv [\dim\ker(1-\sigma)]\nu(2) \pmod 2$.
According to C. Jordan, we have $H_K = \ker(1-\sigma)^r + (1-\sigma)H_K$ for r large. On $\ker(1-\sigma)^r$, the homomorphism σ has only the eigenvalue 1. In particular, $(1-\sigma)H_K \subseteq H_K$ has a complement V such that 2 is the only eigenvalue of $(1+\sigma)|_V$. Hence,

$$\begin{aligned}\nu(\det(1+\sigma)) &= \nu(\det(1+\sigma))|_V + \nu(\det(1+\sigma))|_{(1-\sigma)H_K} \\ &\equiv \nu(\det(1+\sigma))|_V \qquad\qquad \pmod 2 \\ &= \nu(2)[\dim V] \\ &= \nu(2)[\operatorname{rk} H - \dim(1-\sigma)H_K] \\ &= \nu(2)[\dim\ker(1-\sigma)]\,.\end{aligned}$$

Fourth step. Orthogonality.
Finally, σ is orthogonal with respect to a non-degenerate bilinear form. In this situation, it is well-known [42, Example 2.6.C.i.b)] (see also [39, §1.A1, Lemma 4.ii)]) that

$$\dim\ker(1-\sigma) \equiv \dim\ker(1-\sigma)^r \pmod 2$$

for $r \gg 0$.

The right hand side is the algebraic multiplicity of the eigenvalue 1. As the eigenvalues of an orthogonal matrix come in pairs $(z, \frac{1}{z})$ and, by our assumption, (-1) is not an eigenvalue, it has the same parity as $\operatorname{rk} H$. The assertion follows. □

Remarks 2.12. i) There is a conjecture of J.-P. Serre that the operation of Frob on l-adic cohomology is always semisimple. Then, as the eigenvalues come in pairs $(z, \frac{1}{z})$, the congruence $\dim\ker(1-\mathrm{Frob}) \equiv N \pmod 2$ is clear. Thus, conjecturally, the argument on the sizes of the Jordan blocks is not necessary in the application to l-adic cohomology.

ii) Suppose that $q = p^k$ for a prime $p \neq 2$ and let X be a surface such that the canonical sheaf $K \in \operatorname{Pic}(X_{\overline{\mathbb{F}}_q})$ is divisible by 2. Then the case $l = 2$ of Theorem 1.5 may be treated directly.

Indeed, in this situation, Wu's formula [41, Proposition 2.1] implies that $\langle x, x\rangle \in 2\mathbb{Z}_2$ for every $x \in H$. Therefore, by Lemma 2.3.b), $H/(1+\mathrm{Frob})H$ is of even length. Moreover, the assumption enforces that K^2 is even. Hence, $N = \dim H^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_2(1))$ is even, too, by Noether's formula [3, I.14].

2.13. Proof of Theorem 1.9. Here, according to our assumption, we have $p \neq 2$. Again, we may assume without restriction that $\Phi(-1) \neq 0$. In view of Theorem 1.5, it will suffice to prove that $q^{a(X)}\Phi(-1)$ is of even p-adic valuation. Writing $q = p^k$, this means that $ka(X) + \nu_p((1+z_1)\cdots(1+z_N))$ is even.

For this, let $W := W(\mathbb{F}_q)$ be the Witt ring and K its field of fractions. The crystalline cohomology groups $H^d(X/W)$ are finitely generated W-modules, acted upon semilinearly by the absolute Frobenius $\mathbf{F}$. The operation of $\mathbf{F}^k$ is W-linear and such that its characteristic polynomial coincides with $\Phi_0^{(d)}$ [26].

Again, $H := H^d(X/W)/\text{tors}$ is equipped with a natural perfect pairing [4, Ch. VII, Théorème 2.1.3]

$$\langle\,.\,,.\rangle\colon H \times H \longrightarrow W\,.$$

The Frobenius operation is, however, compatible with this pairing only in the sense that $\langle \mathbf{F}(x), \mathbf{F}(y)\rangle = p^d\langle x, y\rangle$ for every $x, y \in H$ [8, Exposé II, Exemple 1.1.ii)]. The map $\sigma := \mathbf{F}/p^{d/2}\colon H_K \to H_K$ respects the pairing $\langle\,.\,,.\rangle$. H is, in fact, a Dieudonné module and thus carries a rich structure [29]. We shall use only a small part of it. Let us distinguish two cases.

First case. k is odd.
Observe that H is a free module as well over the Witt ring $W(\mathbb{F}_p)$ and σ is $W(\mathbb{F}_p)$-linear. Clearly, $\text{rk}_{W(\mathbb{F}_p)} H = k{\cdot}\text{rk}_W H$. The eigenvalues of σ, as a $W(\mathbb{F}_p)$-linear map, are all the k-th roots of the zeroes $z_1, \dots, z_N$ of Φ.

Furthermore, σ is orthogonal with respect to the $\mathrm{Q}_p := \mathrm{Q}(W(\mathbb{F}_p))$-bilinear extension $H_K \times H_K \to \mathrm{Q}_p$ of the perfect pairing $\text{tr}_{W(\mathbb{F}_{p^k})/W(\mathbb{F}_p)} \circ \langle\,.\,,.\rangle\colon H \times H \to W(\mathbb{F}_p)$. The operation of $(-\sigma)$ is orthogonal with respect to the pairing, too. Thus, Corollary 2.7 shows that $\ell_{W(\mathbb{F}_p)}(\sigma(H){+}H/H){+}\nu_{W(\mathbb{F}_p)}(\det(1{+}\sigma))$ is even. By Lemma 2.15, the first summand is equal to $ka(X)$. The second summand is the p-adic valuation of

$$\prod_{i=1}^{N}\prod_{r^k=z_i}(1+r) = \prod_{i=1}^{N}(1+z_i)\,.$$

Second case. k is even.
Here, the argument is slightly more involved. We first observe that H is a free module over the Witt ring $W(\mathbb{F}_{p^{k/2}})$, too, and that $\sigma^{k/2}$ is $W(\mathbb{F}_{p^{k/2}})$-linear. Clearly, $\text{rk}_{W(\mathbb{F}_{p^{k/2}})} H = 2{\cdot}\text{rk}_W H$. Moreover, $\sigma^{k/2}$ is orthogonal with respect to the $\mathrm{Q}_{p^{k/2}} := \mathrm{Q}(W(\mathbb{F}_{p^{k/2}}))$-bilinear extension $H_K \times H_K \to \mathrm{Q}_{p^{k/2}}$ of the perfect pairing $\text{tr}_{W(\mathbb{F}_{p^k})/W(\mathbb{F}_{p^{k/2}})} \circ \langle\,.\,,.\rangle\colon H \times H \to W(\mathbb{F}_{p^{k/2}})$.

Now choose a unit $u \in W(\mathbb{F}_{p^k})$ such that $\text{tr}_{W(\mathbb{F}_{p^k})/W(\mathbb{F}_{p^{k/2}})}(u) = 0$, i.e., such that its conjugate is $(-u)$. Then $\sigma^{k/2}(ux) = -u\sigma^{k/2}(x)$ for all $x \in H$.

We define a $W(\mathbb{F}_{p^{k/2}})$-linear map $T\colon H_K \to H_K$ by

$$T(x) := \sigma^{k/2}(ux)\,.$$

This yields $T \circ T = -u^2\sigma^k$. Furthermore, T is $W(\mathbb{F}_{p^k})$-semilinear and one has $\langle Tx, Ty\rangle = \langle ux, uy\rangle = u^2\langle x, y\rangle$ for all $x, y \in H$.

We see that the eigenvalues of T, as a $W(\mathbb{F}_{p^{k/2}})$-linear map, are all the square roots of the numbers $(-u^2z_1), \ldots, (-u^2z_N)$, for $z_1, \ldots, z_N$ the zeroes of Φ. Moreover, $[T(H) + H]/H$ is of even $W(\mathbb{F}_{p^{k/2}})$-length as it is, in fact, a $W(\mathbb{F}_{p^k})$-module.

Finally, put $\overline{H} := H \otimes_{W(\mathbb{F}_{p^{k/2}})} W(\overline{\mathbb{F}}_p)$ and extend T to a $W(\overline{\mathbb{F}}_p)$-linear map $\overline{T}\colon \overline{H} \to \overline{H}$. Then $\frac{1}{u}\overline{T}$ is orthogonal. On the other hand,

$$[\tfrac{1}{u}\overline{T}(\overline{H}) + \overline{H}]/\overline{H} = [\overline{T}(\overline{H}) + \overline{H}]/\overline{H}\,,$$

as $\frac{1}{u}$ is a unit, and the latter $W(\overline{\mathbb{F}}_p)$-module is of even length. Corollary 2.7 shows that $\nu_{W(\overline{\mathbb{F}}_p)}(\det(1 - \frac{1}{u}\overline{T}))$ is even. This number is nothing but the p-adic valuation of

$$\prod_{i=1}^{N} \prod_{r^2=-z_i} (1-r) = \prod_{i=1}^{N} (1+z_i)\,. \qquad \square$$

Remark 2.14. One might want to prove Theorem 1.9 in characteristic 2 along the same lines as Theorem 1.5. For this, one would need, at least, the theory of Steenrod squares and Wu classes, as well as Wu's formula, for crystalline cohomology of varieties in characteristic 2. It seems, however, that such a theory is not yet available in the literature.

Lemma 2.15. *Let X be a smooth proper variety of even dimension d over $\mathbb{F}_q$, $H := H^d(X/W)/\mathrm{tors}$, and $\sigma = \mathbf{F}/p^{d/2}$. Then $a(X) = \ell_W(\sigma(H) + H/H)$.*

Proof. First, observe that $\sigma(H)$, being the image of an $\mathbf{F}$-semilinear map, is indeed a W-module. Furthermore, we have $H/\mathbf{F}H \cong \bigoplus_{m>0} (W/p^mW)^{h'_{d-m,m}}$. Hence, there is a basis of H such that, under the corresponding isomorphism $H \cong W^N$, one has $\mathbf{F}H \cong \bigoplus_{m\geq 0} p^m W^{h'_{d-m,m}}$. Therefore, $\sigma(H) \cong \bigoplus\limits_{m\geq 0} p^{m-d/2} W^{h'_{d-m,m}}$. The assertion follows. $\square$

Corollary 2.16. *Let X be a smooth proper variety of even dimension d over $\mathbb{F}_q$. Suppose that X is of Hodge-Witt type in degree d, i.e., that the Serre cohomology groups $H^j(X, W\Omega^m_X)$ are finitely generated W-modules for $j + m = d$. Then*

$$a(X) = -\sum_{\nu_q(z_i)<0} \nu_q(z_i)\,.$$

Proof. We have $H^d(X/W) \cong \bigoplus_m H^{d-m,m}$ for $H^{d-m,m} := H^{d-m}(X, W\Omega^m_X)$, as is shown in [25, Théorème IV.4.5]. On $H^{d-m,m}$, $\mathbf{F}$ operates as p^mF for F the usual Frobenius on Serre cohomology. Thus, σ acts as $p^{m-d/2}F$. For $m \geq d/2$, this ensures that the corresponding summand is mapped to H.

Thus, assume that $m < d/2$. On Serre cohomology, there is a second operator, the Verschiebung V, such that $FV = p$. Hence $\sigma p^{d/2-m-1}V = \mathrm{id}$, implying

$$H^{d-m,m} \otimes_W \mathrm{Q}(W) \supseteq \sigma(H^{d-m,m}) \supseteq H^{d-m,m}\,.$$

Lemma 2.15 shows that $a(X) = -\nu_q(\det(\sigma|_{\bigoplus\limits_{m<d/2} H^{d-m,m}}))$, which is equivalent to the assertion. $\square$

3. An application and examples

3.1. An application to the odd-dimensional supersingular case.

Lemma 3.1. *Let p be a prime and d_1 and d_2 two odd integers. Moreover, let $\Phi_1 \in \mathbb{Q}[T]$ and $\Phi_2 \in \mathbb{Q}[T]$ be polynomials of even degrees N_1 and N_2 that fulfill the functional equations*

$$\Phi_1(p^{d_1}/T) = \frac{p^{d_1 N_1/2}}{T^{N_1}}\Phi_1(T) \quad \textit{and} \quad \Phi_2(p^{d_2}/T) = \frac{p^{d_2 N_2/2}}{T^{N_2}}\Phi_2(T)\,.$$

For $z_i^{(1)}$ the zeroes of Φ_1 and $z_j^{(2)}$ the zeroes of Φ_2, let Φ be the monic polynomial with the zeroes $z_i^{(1)} z_j^{(2)}/p^{\frac{d_1+d_2}{2}}$.
Then $p^{N_1 N_2/4}\Phi(-1)$ is a square in $\mathbb{Q}$.

Proof. The assumption implies that the zeroes come in pairs with products p^{d_1} and p^{d_2}, respectively. For two pairs of zeroes, the corresponding four zeroes of Φ are

$$\frac{z_i^{(1)}}{p^{d_1/2}}\cdot\frac{z_j^{(2)}}{p^{d_2/2}}, \quad \frac{z_i^{(1)}}{p^{d_1/2}}\cdot\frac{p^{d_2/2}}{z_j^{(2)}}, \quad \frac{p^{d_1/2}}{z_i^{(1)}}\cdot\frac{z_j^{(2)}}{p^{d_2/2}}, \quad \text{and} \quad \frac{p^{d_1/2}}{z_i^{(1)}}\cdot\frac{p^{d_2/2}}{z_j^{(2)}}\,.$$

We now observe the identity

$$(-1-\tfrac{1}{p}u_1u_2)(-1-u_1/u_2)(-1-u_2/u_1)(-1-p/u_1u_2) = p(\tfrac{u_1}{p}+\tfrac{u_2}{p}+1/u_1+1/u_2)^2\,, \quad (2)$$

which applies, since the four zeroes may rationally be written as

$$\frac{1}{p}\frac{z_i^{(1)}}{p^{(d_1-1)/2}}\cdot\frac{z_j^{(2)}}{p^{(d_2-1)/2}}, \quad \frac{z_i^{(1)}}{p^{(d_1-1)/2}}\cdot\frac{p^{(d_2-1)/2}}{z_j^{(2)}}, \quad \frac{p^{(d_1-1)/2}}{z_i^{(1)}}\cdot\frac{z_j^{(2)}}{p^{(d_2-1)/2}}, \text{ and } p\frac{p^{(d_1-1)/2}}{z_i^{(1)}}\cdot\frac{p^{(d_2-1)/2}}{z_j^{(2)}}\,.$$

It shows that the product $\prod_{i,j}(-1 - z_i^{(1)} z_j^{(2)}/p^{\frac{d_1+d_2}{2}})$ is $p^{N_1N_2/4}$ times a square. Indeed, the sums occurring on the right hand side of (2) form a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant set of $N_1N_2/4$ elements. $\square$

Proposition 3.2. *Let X be a smooth proper variety of odd dimension d over a finite field $\mathbb{F}_{p^k}$ for p a prime and k odd. Suppose that $p > 1 + \dim H^d_{\text{ét}}(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l)$,*

$$\dim H^d_{\text{ét}}(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l) \equiv 2 \pmod 4\,,$$

and that all eigenvalues of Frob *on the cohomology are of p-adic valuation $dk/2$. I.e., that the Newton polygon has constant slope $d/2$.*

Then $\pm\sqrt{-p^{dk}}$ are among the eigenvalues.

Proof. Let C be the base extension of a supersingular elliptic curve defined over $\mathbb{F}_p$. Such do exist by the work of M. Eichler [17], see also [21, Proposition 2.4, together with (1.10) and (1.11)]. The eigenvalues of Frob on $H^1_{\text{ét}}(C_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l)$ are $\pm\sqrt{-p^k}$. Indeed, the assumptions imply $p \geq 5$, and hence $p > 2\sqrt{p}$. The claim follows from Hasse's bound.

For Φ the characteristic polynomial of Frob on

$$V := [H^d_{\text{ét}}(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l) \otimes H^1_{\text{ét}}(C_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l)](\tfrac{d+1}{2})\,,$$

Lemma 3.1 guarantees that $p\Phi(-1)$ is a perfect square. But all eigenvalues of Frob on V are p-adic units. As they are l-adic units for every prime $l \neq p$, too, they must be roots of unity [7, Sec. 18, Lemma 2].

To prove the assertion, we need to show $\Phi(-1) = 0$. Assuming the contrary, we see from Lemma 3.6 that, for some $e \geq 1$, all primitive $2p^e$-th roots of unity must be eigenvalues of Frob on V. As with z, $(-z)$ is an eigenvalue, too, this enforces $\dim V \geq 2(p-1)$, that is $\dim H^d_{\text{ét}}(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l) \geq p-1$, a contradiction. □

Remark 3.3. In principle, the idea behind this proof is to apply Theorem 1.9 to $X \times C$. This is, however, not sufficient as there may be eigenvalues (-1) on the products $[H^{i_1}(X, \mathbb{Q}_l) \otimes H^{i_2}(C, \mathbb{Q}_l)](\frac{d+1}{2})$ for $i_1 + i_2 = d+1$, $i_2 \neq 1$.

Example 3.4. Proposition 3.2 may fail in small characteristic, as is seen from the elliptic curve C over $\mathbb{F}_3$, given by $y^2 = x^3 - x - 1$. Then $\#C(\mathbb{F}_3) = 1$, which shows that C is supersingular. Moreover, the characteristic polynomial of Frob is $T^2 - 3T + 3$. The eigenvalues are the two primitive twelfth roots of unity with positive real part, multiplied by $\sqrt{3}$.

Remark 3.5. When trying to carry over the argument from the proof, it turns out that, on $V := [H^1_{\text{ét}}(C_{\overline{\mathbb{F}}_3}, \mathbb{Q}_l) \times H^1_{\text{ét}}(C_{\overline{\mathbb{F}}_3}, \mathbb{Q}_l)](1)$, there are the primitive sixth roots of unity occurring as eigenvalues, together with 1, which appears twice. Therefore, $\Phi(T) = (T^2 - T + 1)(T-1)^2$ and $\Phi(-1) = 12$.

Alternatively, one might combine C with $C'\colon y^2 = x^3 - x$, which is supersingular having four points. On the product $[H^1_{\text{ét}}(C_{\overline{\mathbb{F}}_3}, \mathbb{Q}_l) \otimes H^1_{\text{ét}}(C'_{\overline{\mathbb{F}}_3}, \mathbb{Q}_l)](1)$, one finds $\Phi(T) = (T^2 - T + 1)(T^2 + T + 1)$ and $\Phi(-1) = 3$.

Lemma 3.6. *Let $\Phi \in \mathbb{Q}[T]$ be a monic polynomial such that all its roots are roots of unity. Assume that $|\Phi(-1)| \neq 0$ is a multiple of a prime number p.*

Then Φ is divisible by the cyclotomic polynomial ϕ_{2p^e}, for some $e \geq 1$. In particular, if $p = 2$ then Φ is divisible by ϕ_{2^e}, for some $e \geq 2$.

Proof. Φ is a product of cyclotomic polynomials ϕ_n. For these, it is well known [34, Section 3] that $\phi_1(-1) = -2$, $\phi_2(-1) = 0$, $\phi_{2p^e}(-1) = p$ for p any prime number and $e \geq 1$, and $\phi_n(-1) = 1$ in all other cases. The assertion follows directly from this. □

3.2. **The even-dimensional supersingular case.**

Proposition 3.7. *Let X be a smooth proper variety of even dimension d over a finite field $\mathbb{F}_{p^k}$ for a prime $p \neq 2$ and k odd. Suppose that $a(X) \equiv 1 \pmod 2$ and that all eigenvalues of* Frob *on $H^d_{\text{ét}}(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l(d/2))$ are p-adic units. I.e., that the Newton polygon has constant slope $d/2$.*

a) *Then* (-1) *is an eigenvalue or, for some* $e \geq 1$*, the primitive* $2p^e$*-th roots of unity are eigenvalues.*

b) *If* $p > \dim H^d_{\text{ét}}(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l(d/2)) + 1$ *or* $p > \dim H^d_{\text{ét}}(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l(d/2))$ *and* X *is projective then* (-1) *is an eigenvalue.*

Proof. The eigenvalues are l-adic units, too, for every prime number $l \neq p$, hence they are roots of unity. Theorem 1.9 ensures that $p\Phi(-1)$ is a perfect square. Applying Lemma 3.6 again, we immediately obtain assertion a).

For b), assume the contrary. Then, as eigenvalues, we have the $(p-1)p^{e-1} \geq p-1$ primitive $2p^e$-th roots of unity and, in the projective case, the number 1. Thus, altogether, there are at least $p-1$, respectively p, of them. □

Corollary 3.8. *Let* X *be a supersingular* $K3$ *surface over a finite field* $\mathbb{F}_{p^k}$ *for* $p > 19$ *a prime and* k *odd. Then* (-1) *is an eigenvalue of* Frob *on* $H^2_{\text{ét}}(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l(1))$.

Proof. For $K3$ surfaces, the Hodge spectral sequence degenerates at E_1 [13, Proposition 1.1.a)] and, hence, the conjugate spectral sequence degenerates at E_2 [5, Lemma 8.27.2]. Moreover, all $H^i(X/W)$ are torsion-free ([24, II.7.2] or [13, Proposition 1.1.c)]). Consequently, we have $a(X) = \dim H^2(X, \mathscr{O}_X) = 1$. Cf. Remark 1.8.b). The claim now follows from Proposition 3.7.b). □

Remarks 3.9. i) Corollary 3.8 refines the observation of M. Artin [1, 6.8] that the field of definition of the rank-22 Picard group always contains $\mathbb{F}_{p^2}$.

ii) More generally, let X be any $K3$ surface over a finite field $\mathbb{F}_{p^k}$ for $p \neq 2$ a prime. Theorem 1.9 then asserts that, for $z_1, \ldots, z_{22}$ the eigenvalues of Frob on $H^2_{\text{ét}}(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l(1))$, the expression $p^k\Phi(-1) = p^k(1+z_1)\ldots(1+z_{22})$ is always a square in $\mathbb{Q}$.

iii) Corollary 3.8 is clearly false in small characteristic, as may be seen from the example below.

Examples 3.10. i) For C the supersingular elliptic curve over $\mathbb{F}_3$ from Example 3.4, put $X := \text{Kum}(C \times C)$. Then

$$\Phi(T) = (T^2 - T + 1)(T^2 + T + 1)^5(T-1)^{10}.$$

In particular, we have $\Phi(-1) = 3 \cdot 2^{10}$, in agreement with Remark 3.9.ii).

Indeed, we saw that the characteristic polynomial of Frob on $H^2_{\text{ét}}((C \times C)_{\overline{\mathbb{F}}_3}, \mathbb{Q}_l(1))$ is $(T^2 - T + 1)(T-1)^4$. Furthermore, the 16 two-torsion points form five orbits of size three together with the origin.

ii) Let $X := \text{Kum}(C \times C')$ be the Kummer surface associated to the product of the two supersingular elliptic curves over $\mathbb{F}_3$, considered in Remark 3.5. Then, once again, $\Phi(T) = (T^2 - T + 1)(T^2 + T + 1)^5(T-1)^{10}$.

In fact, we saw that the characteristic polynomial of Frob on $H^2_{\text{ét}}((C \times C')_{\overline{\mathbb{F}}_3}, \mathbb{Q}_l(1))$ is $(T^2 - T + 1)(T^2 + T + 1)(T-1)^2$. In addition, four two-torsion points are defined over the base field, while the others form four orbits of size three.

3.3. **Surfaces. The Artin-Tate formula.** For surfaces, the assertion of Theorem 1.5 is implied by the Tate conjecture. More precisely,

Proposition 3.11. *Let X be a smooth projective surface over a finite field $\mathbb{F}_q$ of characteristic p and let $\Phi = \Phi_1^{(2)} \in \mathbb{Q}[T]$ be the characteristic polynomial of* Frob *on $H^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l(1))$. Put $N := \deg \Phi$ and*

$$\alpha(X) := \dim H^2(X, \mathscr{O}_X) - \dim H^1(X, \mathscr{O}_X) + \tfrac{1}{2} \dim H^1_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l) \,.$$

Suppose that the Tate conjecture is true for X.

Then $(-2)^N q^{\alpha(X)} \Phi(-1)$ is a square in $\mathbb{Q}$.

Proof. Denote the zeroes of Φ, i.e. the eigenvalues of Frob, by $z_1, \ldots, z_\varrho = 1$, $z_{\varrho+1}, \ldots, z_N \neq 1$. If $\Phi(-1) = 0$ then the assertion is true, trivially. Thus, let us suppose the contrary from now on. Then the zeroes $z_i \neq 1$ come in pairs of complex conjugate numbers. In particular, $N - \varrho$ is even.

Furthermore, Frob and Frob^2 have the eigenvalue 1 with the same multiplicity. Hence, the Tate conjecture predicts the rank of $\text{Pic}(X_{\mathbb{F}_{q^2}})$ not to be higher than that of $\text{Pic}(X)$. This shows that $X_{\mathbb{F}_{q^2}}$, too, fulfills the Tate conjecture.

We are therefore in a situation where the Artin-Tate formula [32, Theorem 6.1] computes the discriminants of the Picard lattices $\text{Pic}(X)$ and $\text{Pic}(X_{\mathbb{F}_{q^2}})$, at least up to square factors. The results are

$$(-1)^{\varrho-1} q^{\alpha(X)} \prod_{i=\varrho+1}^{N} (1 - z_i) \quad \text{and} \quad (-1)^{\varrho-1} q^{2\alpha(X)} \prod_{i=\varrho+1}^{N} (1 - z_i^2) \,.$$

Moreover, equality of the ranks implies that $\operatorname{disc} \text{Pic}(X) / \operatorname{disc} \text{Pic}(X_{\mathbb{F}_{q^2}})$ is a necessarily perfect square. This is a standard observation from the theory of lattices. We conclude that $q^{\alpha(X)} \prod_{i=\varrho+1}^{N} (1 + z_i)$ is a square in $\mathbb{Q}$.

On the other hand, $(-2)^N q^{\alpha(X)} \Phi(-1) = 2^{N+\varrho} q^{\alpha(X)} \prod_{i=\varrho+1}^{N} (1 + z_i)$ such that the assertion follows from the fact that $\varrho \equiv N \pmod 2$. □

Remarks 3.12. i) The Artin-Tate formula appears to us as a very natural consequence of the Tate conjecture and the cohomological machinery. Thus, we find it very astonishing that it has the potential to produce incompatible results for a variety and its base extension.

Of course, this does not happen for polynomials that really occur as the characteristic polynomial of the Frobenius on a certain variety. But it occurs for polynomials that otherwise look plausible. This observation was actually the starting point of our investigations.

ii) One might want to compare the Picard lattice of X with that of X_{q^n} for $n > 2$. But this leads to nothing new [18, Corollary 18.i)].

iii) Suppose $\dim H^1(X, \mathscr{O}_X) = \frac{1}{2} \dim H^1_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$ and that X fulfills the assumptions of Remark 1.8.b). Then $\alpha(X) = a(X) = \dim H^2(X, \mathscr{O}_X)$. We do not know how closely Milne's invariant $\alpha(X)$ and our invariant $a(X)$ are related for "pathological" surfaces.

Example 3.13. Let X be the double cover of $\mathbf{P}^2_{\mathbb{F}_7}$, given by

$$w^2 = 6x^6 + 6x^5y + 2x^5z + 6x^4y^2 + 5x^4z^2 + 5x^3y^3 + x^2y^4 + 6xy^5 + 5xz^5 + 3y^6 + 5z^6\,.$$

This is a $K3$ surface of degree two.

The numbers of points on X over the finite fields $\mathbb{F}_7, \ldots, \mathbb{F}_{7^{10}}$ are 60, 2 488, 118 587, 5 765 828, 282 498 600, 13 841 656 159, 678 225 676 496, 33 232 936 342 644, 1 628 413 665 268 026, and 79 792 266 679 604 918.

For the characteristic polynomial of Frob on $H^2_{\text{ét}}(X_{\overline{\mathbb{F}}_7}, \mathbb{Q}_l(1))$, this information leaves us with two candidates, one for each sign in the functional equation,

$$\begin{aligned}\Phi_i(t) = \frac{1}{7}\big(&7t^{22} - 10t^{21} + t^{20} - t^{19} + 6t^{18} - 3t^{17} - 2t^{16} + 4t^{14} - t^{13} - t^{12} \\ &+ (-1)^i(-t^{10} - t^9 + 4t^8 - 2t^6 - 3t^5 + 6t^4 - t^3 + t^2 - 10t + 7)\big)\end{aligned}$$

for $i = 0, 1$. All roots are of absolute value 1.

However, $\Phi_0(-1) = 60/7$ and $(-2)^N 7^{a(X)}\Phi_0(-1) = 2^{24}{\cdot}3{\cdot}5$ is a non-square, which contradicts Theorem 1.5. Therefore, Φ_1 is the characteristic polynomial of Frob on $H^2_{\text{ét}}(X_{\overline{\mathbb{F}}_7}, \mathbb{Q}_l(1))$. The minus sign holds in the functional equation.

Remark 3.14. Alternatively, we may argue as follows. Assume Φ_0 is the characteristic polynomial. Then $\operatorname{rk}\operatorname{Pic}(X_{\mathbb{F}_{49}}) = \operatorname{rk}\operatorname{Pic}(X) = 2$. Indeed, the Tate conjecture is proven for $K3$ surfaces in characteristic ≥ 3 [9, Corollary 2], [35, Theorem 1], cf. [27]. The Artin-Tate formula states that $\operatorname{disc}\operatorname{Pic}(X_{\mathbb{F}_{49}}) \in (-465)(\mathbb{Q}^*)^2$ and $\operatorname{disc}\operatorname{Pic}(X) \in (-31)(\mathbb{Q}^*)^2$. As $\frac{-465}{-31} = 15$ is a non-square, this is contradictory.

3.4. **Cubic fourfolds.**

Example 3.15. Let X be the subvariety of $\mathbf{P}^5_{\mathbb{F}_2}$, given by

$$\begin{aligned}&x_0^3 + x_0^2x_1 + x_0^2x_4 + x_0^2x_5 + x_0x_1x_2 + x_0x_1x_3 + x_0x_1x_4 + x_0x_2x_3 + x_0x_2x_4 \\ &+ x_0x_3x_4 + x_0x_4^2 + x_0x_4x_5 + x_0x_5^2 + x_1^3 + x_1x_2^2 + x_1x_2x_4 + x_1x_2x_5 + x_1x_3^2 \\ &\quad + x_1x_3x_5 + x_1x_4^2 + x_1x_4x_5 + x_2^3 + x_2^2x_5 + x_2x_3^2 + x_3^2x_4 + x_3^2x_5 + x_3x_4^2 \\ &\qquad + x_3x_5^2 + x_4^3 + x_4^2x_5 + x_4x_5^2 + x_5^3 = 0\,.\end{aligned}$$

This is a smooth cubic fourfold. We have $\dim H^4(X, \mathscr{O}_X) = 0$, $\dim H^3(X, \Omega^1_X) = 1$, and $\dim H^2(X, \Omega^2_X) = 21$, According to Remark 1.8.b), this shows $N = 23$ and $a(X) = 1$.

The numbers of points on X over the finite fields $\mathbb{F}_2, \ldots, \mathbb{F}_{2^{11}}$ are 33, 361, 4 545, 69 665, 1 084 673, 17 044 609, 270 543 873, 4 311 990 785, 68 853 026 817, 1 100 586 076 161, and 17 600 769 409 025. The characteristic polynomial of Frob on $H^4_{\text{ét}}(X_{\overline{\mathbb{F}}_2}, \mathbb{Q}_l(2))$ is

$$\begin{aligned}\Phi(t) = \frac{1}{2}(t-1)(&2t^{22} - t^{21} - t^{20} + 2t^{19} - 2t^{17} + t^{16} + t^{15} - 2t^{14} + t^{13} + t^{12} \\ &- t^{11} + t^{10} + t^9 - 2t^8 + t^7 + t^6 - 2t^5 + 2t^3 - t^2 - t + 2)\,.\end{aligned}$$

It turns out that $\Phi(-1) = -1$, in agreement with Theorem 1.5. Observe that, in this example, the assertion of Theorem 1.9 is true, albeit the characteristic of the base field is 2.

Remark 3.16. The degree 22 factor of Φ is irreducible over $\mathbb{Q}$. In particular, X is certainly not special in the sense of B. Hassett [22].

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