Cubic surfaces with a Galois invariant pair of Steiner trihedra

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Abstract

We present a method to construct non-singular cubic surfaces over $\mathbb{Q}$ with a Galois invariant pair of Steiner trihedra. We start with cubic surfaces in a form generalizing that of A. Cayley and G. Salmon. For these, we develop an explicit version of Galois descent.

1 Introduction

1.1. —— The configuration of the 27 lines upon a smooth cubic surface is highly symmetric. The group of all permutations respecting the intersection pairing is isomorphic to the Weyl group $W(E_6)$ of order 51 840.

When S is a cubic surface over $\mathbb{Q}$, the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ operates on the 27 lines. This yields a subgroup $G \subseteq W(E_6)$.

1.2. —— There are exactly 350 conjugacy classes of subgroups of $W(E_6)$. Only for about one half of them, explicit examples of cubic surfaces over $\mathbb{Q}$ are known.

General cubic surfaces [EJ1] lead to the full $W(E_6)$. In [EJ2], we constructed examples for the index two subgroup which is the simple group of order 25 920. Other examples may be obtained by fixing a $\mathbb{Q}$-rational line or tritangent plane. Generically, this yields the maximal subgroups in $W(E_6)$ of indices 27 and 45, respectively. It is not yet clear which smaller groups arise by further specialization.

On the other hand, there are a number of rather small subgroups in $W(E_6)$ for which examples may be constructed easily. Blowing up six points in $\mathbf{P}^2_{\mathbb{Q}}$ forming a

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Galois invariant set leads to a cubic surface with a Galois invariant sixer. It is clear that examples for all the 56 corresponding conjugacy classes of subgroups may be constructed in this way. There are a few more trivial cases, e.g. diagonal surfaces, but all in all not more than 70 of the 350 conjugacy classes of subgroups may be realized by such elementary methods.

In [EJ3], we presented a method to construct cubic surfaces over $\mathbb{Q}$ with a Galois invariant double-six. A simple calculation in `GAP` shows that there are 102 conjugacy classes of subgroups of $W(E_6)$ fixing a double-six but no sixer. For each of them, explicit examples of cubic surfaces are given in the list [EJ3a]. The most interesting ones were reproduced in [EJ3].

1.3. —— In this article, we present a method to construct cubic surfaces over $\mathbb{Q}$ with a Galois invariant pair of Steiner trihedra. Our method is based on cubic surfaces in a form slightly generalizing that of A. Cayley and G. Salmon. For these, we develop an explicit version of Galois descent.

There are 63 conjugacy classes of subgroups of $W(E_6)$ which fix a pair of Steiner trihedra but no double-six. We constructed explicit examples of cubic surfaces for each of them. Some of the most interesting ones will be presented in the final section.

2 Steiner trihedra

This section will mainly recall definitions and facts which are necessary for the sequel. Most of them were known to the geometers of the 19th century [St, Do].

2.1. —— Let S be a smooth cubic surface over an algebraically closed field. It is well-known that S contains a total of 27 lines. There are exactly 45 planes cutting three lines out of S. These are called the *tritangent planes*.

Two planes in $\mathbf{P}^3$ which are different from each other always meet in a single line. Given two tritangent planes, there are two possibilities. Their intersection is either one of the 27 lines contained in S or a line not contained in S. For a tritangent plane E, there are twelve tritangent planes meeting E within the surface, four for each of the lines in $E \cap S$. 32 tritangent planes remain which meet E otherwise.

2.2. Remark. —— The set of pairs of distinct tritangent planes is acted upon by the automorphism group $W(E_6)$. Under this operation, that set is decomposed into exactly two orbits according to the way the tritangent planes meet each other.

2.3. Definitions. —— a) A *trihedron* consists of three distinct tritangent planes such that the intersection of any two is not contained in S.

b) For a trihedron $\{E_1, E_2, E_3\}$, a plane E is called a *conjugate plane* if each of the lines $E_1 \cap E$, $E_2 \cap E$, and $E_3 \cap E$ is contained in the surface S.

2.4. Fact-Definition. —— A trihedron may have either no, exactly one, or exactly three conjugate planes. Correspondingly, a trihedron is said to be of the *first kind, second kind,* or *third kind.* Trihedra of the third kind are also called *Steiner trihedra.*

2.5. Remark. —— Let two tritangent planes E_1, E_2 be given such that their intersection line is not contained in the surface S. Then, there are three tritangent planes meeting both E_1 and E_2 in lines within S. Nine further tritangent planes meet E_1 on S. Analogously, nine tritangent planes only meet E_2 within S.

22 tritangent planes remain. Twelve of them complete $\{E_1, E_2\}$ to a trihedron of the first kind. For nine tritangent planes E, $\{E_1, E_2, E\}$ becomes a trihedron of the second kind. Finally, there is a unique tritangent plane such that $\{E_1, E_2, E\}$ is a Steiner trihedron.

Consequently, on a smooth cubic surface, there are 2880 trihedra of the first kind, 2160 trihedra of the second kind, and 240 Steiner trihedra. The group $W(E_6)$ acts transitively on the set of all Steiner trihedra. In fact, the operations on trihedra of the first and second kinds are transitive, too.

2.6. Fact. —— a) *Steiner trihedra come in pairs. Actually, the three conjugate planes of a Steiner trihedron form another Steiner trihedron.*

b) *Two trihedra define the same sets of lines if and only if they form a pair of Steiner trihedra.*

c) *The nine lines defined by a Steiner trihedron form the complement of the lines contained in a triple of azygetic double-sixes.*

Proof. Recall that two double-sixes on a non-singular cubic surface may be either syzygetic or azygetic according to the number of lines they have in common. Further, a pair of azygetic double-sixes uniquely determines a third double-six, azygetic to both of them [Do, EJ4].

The assertion itself may best be seen in the blown-up model. In Schläfli's notation [Sch, p. 116], one of the Steiner trihedra is formed by the tritangent planes $[c_{14}, c_{25}, c_{36}]$, $[c_{15}, c_{26}, c_{34}]$, and $[c_{16}, c_{24}, c_{35}]$. Indeed, the three conjugate planes are given by $[c_{14}, c_{26}, c_{35}]$, $[c_{15}, c_{24}, c_{36}]$, and $[c_{16}, c_{25}, c_{34}]$. Further, the "standard" triple of azygetic double-sixes

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{pmatrix}, \quad \begin{pmatrix} a_1 & a_2 & a_3 & c_{56} & c_{46} & c_{45} \\ c_{23} & c_{13} & c_{12} & b_4 & b_5 & b_6 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} c_{23} & c_{13} & c_{12} & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & c_{56} & c_{46} & c_{45} \end{pmatrix}$$

is exactly formed by the remaining lines. □

2.7. Notation. —— Let $l_1, \ldots l_9$ be the nine lines defined by a Steiner trihedron. Then, we will denote the corresponding pair of Steiner trihedra by a rectangular symbol of the form

$$\begin{bmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \\ l_7 & l_8 & l_9 \end{bmatrix}.$$

The planes of the trihedra contain the lines noticed in the rows and columns.

2.8. Proposition. — *Let a triple of azygetic double-sixes be given.*

a) *Then, of the corresponding six sixers, one may form fifteen pairs. Nine of them are disjoint. The other six intersections are mutually disjoint triplets $D_0, \dots, D_5$.*

b) *These may be arranged in a diagram of the form*

$$\begin{array}{cccc} & D_2 & D_1 & \\ D_3 & & & D_0 \\ & D_4 & D_5 & \end{array}$$

such that two triplets combine to a sixer if and only if they are adjacent. Such a diagram is unique up to rotation and reflection.

c) *Further, the following properties may be read off the diagram.*

i) *Every line in D_i meets every line in D_j if and only if D_i and D_j are opposite.*

ii) *Two sixers form a double-six if and only if they are opposite.*

iii) *The nine lines in $D_0 \cup D_2 \cup D_4$ are defined by a Steiner trihedron. Analogously, for the nine lines in $D_1 \cup D_3 \cup D_5$.*

Proof. Again, let us work in the blown-up model and consider the standard triple of azgetic double-sixes formed by the 18 lines $a_1, \dots, a_6$, $b_1, \dots, b_6$, c_{12}, c_{13}, c_{23}, c_{45}, c_{46}, and c_{56}.

Then, a) is immediately verified. The six triplets which appear as intersections of the sixers are $A_l := \{a_1, a_2, a_3\}$, $A_r := \{a_4, a_5, a_6\}$, $B_l := \{b_1, b_2, b_3\}$, $B_r := \{b_4, b_5, b_6\}$, $C_l := \{c_{12}, c_{13}, c_{23}\}$, and $C_r := \{c_{45}, c_{46}, c_{56}\}$.

b) Consider the diagram

$$\begin{array}{cccc} & A_l & A_r & \\ C_r & & & C_l\,. \\ & B_l & B_r & \end{array}$$

The property stated is directly checked. Uniqueness is clear.

c) Properties i) and ii) may be verified immediately. Further, we have the two pairs

$$\begin{bmatrix} a_1 & b_2 & c_{12} \\ b_3 & c_{23} & a_2 \\ c_{13} & a_3 & b_1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} a_4 & b_5 & c_{45} \\ b_6 & c_{56} & a_5 \\ c_{46} & a_6 & b_4 \end{bmatrix}$$

of Steiner trihedra. □

2.9. Fact. —— *Given a pair of Steiner trihedra, there is a unique way to decompose the 18 remaining lines into two sets of nine such that both are defined by Steiner trihedra.*

Proof. The *existence* of a decomposition as desired follows from Fact 2.6.c) and Proposition 2.8.c.iii). To see *uniqueness*, we need an overview over all 120 pairs of Steiner trihedra. In the blown-up model, these are of the types

$$\begin{bmatrix} a_i & b_j & c_{ij} \\ b_k & c_{jk} & a_j \\ c_{ik} & a_k & b_i \end{bmatrix}, \qquad \begin{bmatrix} c_{il} & c_{jm} & c_{kn} \\ c_{jn} & c_{kl} & c_{im} \\ c_{km} & c_{in} & c_{jl} \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} a_i & b_j & c_{ij} \\ b_k & a_l & c_{kl} \\ c_{ik} & c_{jl} & c_{mn} \end{bmatrix}.$$

We have 20 pairs of Steiner trihedra of the first type, 10 of the second, and 90 of the last type. Having seen this, it is easy to verify that there are exactly two pairs of Steiner trihedra having no line in common with

$$\begin{bmatrix} c_{14} & c_{25} & c_{36} \\ c_{26} & c_{34} & c_{15} \\ c_{35} & c_{16} & c_{24} \end{bmatrix}.$$

□

2.10. Definition. —— Given a pair of Steiner trihedra, we will call the two other pairs *complementary* to the given one if, altogether, they define all the 27 lines.

2.11. Remarks. —— i) The investigation above shows, in fact, that, for each pair of Steiner trihedra, there are exactly two pairs having no line in common, 54 pairs having two lines in common, 36 pairs having three lines in common, and 27 pairs which have five lines in common with the nine lines defined by the pair given.

ii) The subgroup of $W(E_6)$ stabilizing a pair of Steiner trihedra is isomorphic to $[(S_3 \times S_3) \rtimes \mathbb{Z}/2\mathbb{Z}] \times S_3$ of order 432.

A calculation in `GAP` shows that, indeed, this group operates on pairs of Steiner trihedra such that the orbits have lengths 1, 2, 27, 36, and 54.

3 The generalized Cayley-Salmon form

3.1. Notation. —— One way to write down a cubic surface explicitly is the so-called *Cayley-Salmon form* [Do, §9.3]. A slight generalization is the following. For $u_0, u_1 \neq 0$, denote by $S_{u_0,u_1}^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}$ the cubic surface given in $\mathbf{P}^5$ by the system of equations

$$\begin{aligned} u_0X_0X_1X_2 + u_1X_3X_4X_5 &= 0\,, \\ a_0X_0 + a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5 &= 0\,, \\ b_0X_0 + b_1X_1 + b_2X_2 + b_3X_3 + b_4X_4 + b_5X_5 &= 0\,. \end{aligned}$$

3.2. Remark. —— The geometric meaning of these equations is to intersect the cubic fourfold given by $u_0X_0X_1X_2+u_1X_3X_4X_5=0$ with two hyperplanes. All these fourfolds are actually isomorphic to each other. For $u_0 = u_1 = 1$, the classical Cayley-Salmon form is obtained.

3.3. Definition. —— Let $S^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}_{u_0,u_1}$ be a cubic surface in generalized Cayley-Salmon form. We will call the general cubic polynomial

$$\Phi^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}_{u_0,u_1}(T) := \frac{1}{u_0}(a_0+b_0T)(a_1+b_1T)(a_2+b_2T) - \frac{1}{u_1}(a_3+b_3T)(a_4+b_4T)(a_5+b_5T)$$

the *auxiliary polynomial* associated with $S^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}_{u_0,u_1}$. We will simply write Φ instead of $\Phi^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}_{u_0,u_1}$ when there is no danger of confusion.

3.4. Proposition (The discriminantal locus)**.** —— *Over a base field K of characteristic $\neq 3$, the cubic surface $S^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}_{u_0,u_1}$ is singular if and only if*

i)

$$\det\begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix} = 0$$

for some $i \in \{0,1,2\}$ and $j \in \{3,4,5\}$, or

ii) *the discriminant of the auxiliary polynomial vanishes.*

Proof. There are two ways the intersection of the cubic fourfold given by

$$u_0X_0X_1X_2 + u_1X_3X_4X_5 = 0 \tag{1}$$

with the two hyperplanes may become singular. On one hand, it might happen that both hyperplanes meet a singular point of the fourfold.

The singular locus of (1) is given by

$$X_0X_1 = X_0X_2 = X_1X_2 = X_3X_4 = X_3X_5 = X_4X_5 = 0.$$

This means nothing but $X_{i_1} = X_{i_2} = X_{j_1} = X_{j_2} = 0$ for $i_1 \neq i_2 \in \{0,1,2\}$ and $j_1 \neq j_2 \in \{3,4,5\}$. We meet such a point if and only if the corresponding determinantal condition is fulfilled. The degenerate case that the two linear forms are lineraly dependent is covered by this case, too.

On the other hand, the hyperplanes might meet the fourfold tangentially in a certain point $(x_0 : \ldots : x_5)$. This means that the tangent hyperplane of the fourfold at $(x_0 : \ldots : x_5)$ is a linear combination of the two hyperplanes given. The tangent hyperplane is given by

$$u_0(x_1x_2X_0 + x_0x_2X_1 + x_0x_1X_2) + u_1(x_4x_5X_3 + x_3x_5X_4 + x_3x_4X_5) = 0\,.$$

There are two cases.

First Case: One of the coordinates $x_0, \ldots, x_5$ vanishes.

Then, in both summands of (1), at least one factor must vanish. Without restriction, suppose $x_0 = x_3 = 0$. The tangent hyperplane is then given by $MX_0 + NX_3 = 0$ for certain constants M and N. This may be a linear combination of $a_0X_0+a_1X_1+a_2X_2+a_3X_3+a_4X_4+a_5X_5$ and $b_0X_0+b_1X_1+b_2X_2+b_3X_3+b_4X_4+b_5X_5$ only if $\det(\begin{smallmatrix} a_2 & a_5 \\ b_2 & b_5 \end{smallmatrix}) = 0$.

Second Case: $x_0, \ldots, x_5 \neq 0$.

Let the tangent hyperplane be given by $M_0X_0 + \ldots + M_5X_5 = 0$. Then, necessarily, $\frac{1}{u_0}M_0M_1M_2 = \frac{1}{u_1}M_3M_4M_5$. The point of tangency is

$$\begin{aligned}
&(x_0 : \ldots : x_5) = \\
&\qquad \left(\tfrac{1}{u_0}M_1M_2 : \tfrac{1}{u_0}M_0M_2 : \tfrac{1}{u_0}M_0M_1 : (-\tfrac{1}{u_1}M_4M_5) : (-\tfrac{1}{u_1}M_3M_5) : (-\tfrac{1}{u_1}M_3M_4)\right).
\end{aligned}$$

We suppose that the tangent hyperplane is a linear combination of the two linear forms given. Then, $M_i = b_i$ or $M_i = a_i + b_it$, for some t, $i = 0, \ldots, 5$. The first variant may be interpreted as "$t = \infty$".

The conditions that $(x_0 : \ldots : x_5)$ must be contained in both given hyperplanes may be rephrased as

$$\begin{aligned}
&\tfrac{1}{u_0}[a_0(a_1 + b_1t)(a_2 + b_2t) + a_1(a_0 + b_0t)(a_2 + b_2t) + a_2(a_0 + b_0t)(a_1 + b_1t)] \\
&\quad - \tfrac{1}{u_1}[a_3(a_4 + b_4t)(a_5 + b_5t) - a_4(a_3 + b_3t)(a_5 + b_5t) - a_5(a_3 + b_3t)(a_4 + b_4t)] = 0
\end{aligned}$$

and

$$\begin{aligned}
&\tfrac{1}{u_0}[b_0(a_1 + b_1t)(a_2 + b_2t) + b_1(a_0 + b_0t)(a_2 + b_2t) + b_2(a_0 + b_0t)(a_1 + b_1t)] \\
&\quad - \tfrac{1}{u_1}[b_3(a_4 + b_4t)(a_5 + b_5t) - b_4(a_3 + b_3t)(a_5 + b_5t) - b_5(a_3 + b_3t)(a_4 + b_4t)] = 0\,.
\end{aligned}$$

In terms of the auxiliary polynomial, these two quadratic polynomials are $3\Phi - t\Phi'$ and Φ'. As they have a common zero, we see that $\mathrm{Res}_{2,2}(3\Phi - t\Phi', \Phi')$ must vanish.

Let us calculate this resultant. First, the leading coefficient of Φ' is equal to $3(\frac{1}{u_0}b_0b_1b_2 - \frac{1}{u_1}b_3b_4b_5)$. Hence, according to the definition of the resultant,

$$\mathrm{Res}_{2,2}(3\Phi - t\Phi', \Phi') = \frac{\mathrm{Res}_{3,2}(3\Phi - t\Phi', \Phi')}{3(\frac{1}{u_0}b_0b_1b_2 - \frac{1}{u_1}b_3b_4b_5)}\,.$$

On the other hand, as $t\Phi'$ is a multiple of Φ',

$$\begin{aligned}
&\mathrm{Res}_{3,2}(3\Phi - t\Phi', \Phi') = \mathrm{Res}_{3,2}(3\Phi, \Phi') = \\
&\qquad\qquad 9\,\mathrm{Res}_{3,2}(\Phi, \Phi') = -9(\tfrac{1}{u_0}b_0b_1b_2 - \tfrac{1}{u_1}b_3b_4b_5)\,\mathrm{disc}(\Phi)\,.
\end{aligned}$$

Consequently, $\mathrm{Res}_{2,2}(3\Phi - t\Phi', \Phi') = -3\,\mathrm{disc}(\Phi)$.

For u_0 and u_1 fixed, this is an irreducible polynomial in twelve variables. The corresponding component really occurs in the discriminantal variety as, for example,

$$\begin{aligned} u_0x_0 + x_1 + x_2 + u_1x_3 - x_4 - x_5 &= 0 \\ 2u_1x_3 + x_4 + x_5 &= 0 \end{aligned}$$

yields tangency at $(\frac{1}{u_0} : 1 : 1 : (-\frac{1}{u_1}) : 1 : 1)$ although we do not meet any of the nine determinantal components. $\square$

3.5. Remark. —— The actual discriminant is a polynomial of degree 32 in u_0 and u_1 and bidegree $(24, 24)$ in the a's and b's. It factors into the squares of the nine determinants $\det(\begin{smallmatrix} a_i & a_j \\ b_i & b_j \end{smallmatrix})$ and $u_0^{18}u_1^{18} \operatorname{disc} \Phi$. The necessity of taking the squares is motivated by [EJ2, Theorem 2.12] and Corollary 6.8, below.

4 Obvious and non-obvious lines

4.1. —— Let $S = S_{u_0,u_1}^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}$ be a non-singular cubic surface in generalized Cayley-Salmon form. Then, on S, there are nine lines of the type

$$L_{ij} : X_i = X_j = 0, \qquad i = 0, 1, 2,\ j = 3, 4, 5,$$

which we call the *obvious lines.*

4.2. Fact. —— *The linear forms $X_0, \ldots, X_5$ define six tritangent planes $E_0, \ldots, E_5$. They form a pair*

$$\big(\{E_0, E_1, E_2\}, \{E_3, E_4, E_5\}\big)$$

of Steiner trihedra.

4.3. Remark. —— The generalized Cayley-Salmon form therefore distinguishes one of the 120 pairs of Steiner trihedra.

4.4. Remark. —— The situation here is analogous to that of a non-singular cubic surface in the hexahedral form of L. Cremona [Cr] and Th. Reye [Re]. A cubic surface in hexahedral form is given in $\mathbf{P}^5$ by a system of equations of the type

$$\begin{aligned} X_0^3 + \quad X_1^3 + \quad X_2^3 + \quad X_3^3 + \quad X_4^3 + \quad X_5^3 &= 0\,, \\ X_0 + \quad X_1 + \quad X_2 + \quad X_3 + \quad X_4 + \quad X_5 &= 0\,, \\ a_0X_0 + a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5 &= 0\,. \end{aligned}$$

Here, there are the 15 obvious lines given by $X_{i_0} + X_{i_1} = X_{i_2} + X_{i_3} = 0$ for $(i_0i_1)(i_2i_3)(i_4i_5)$ a partition of the set $\{0, \ldots, 5\}$.

4.5. Proposition. —— *Let K be a field s.t.* $\mathrm{char}\, K \neq 3$, *$S = S_{u_0,u_1}^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}$ a non-singular cubic surface in generalized Cayley-Salmon form over K, and λ a zero of the auxiliary polynomial Φ associated with S.*

a) *Then, S has a hexahedral form in the coordinates*

$$\begin{aligned} Z_0 &:= -Y_0 + Y_1 + Y_2, \quad Z_1 := Y_0 - Y_1 + Y_2, \quad Z_2 := Y_0 + Y_1 - Y_2, \\ Z_3 &:= -Y_3 + Y_4 + Y_5, \quad Z_4 := Y_3 - Y_4 + Y_5, \quad Z_5 := Y_3 + Y_4 - Y_5 \end{aligned}$$

for

$$Y_i := (a_i + b_i\lambda)X_i, \quad i = 0, \ldots, 5\,.$$

b) *In particular, six non-obvious lines on S may be described by*

$$L_\rho^\lambda : Z_0 + Z_{\rho(0)} = Z_1 + Z_{\rho(1)} = 0$$

for $\rho\colon \{0,1,2\} \to \{3,4,5\}$ a bijection.

Proof. a) We have

$$\tfrac{1}{u_0}(a_0 + b_0\lambda)(a_1 + b_1\lambda)(a_2 + b_2\lambda) = \tfrac{1}{u_1}(a_3 + b_3\lambda)(a_4 + b_4\lambda)(a_5 + b_5\lambda)\,.$$

Hence, S is given by

$$Y_0Y_1Y_2 + Y_3Y_4Y_5 = 0\,, \tag{2}$$
$$Y_0 + Y_1 + Y_2 + Y_3 + Y_4 + Y_5 = 0\,, \tag{3}$$

and another linear relation. (2) and (3) together imply

$$\begin{aligned} &(-Y_0 + Y_1 + Y_2)^3 + (Y_0 - Y_1 + Y_2)^3 + (Y_0 + Y_1 - Y_2)^3 \\ &\qquad + (-Y_3 + Y_4 + Y_5)^3 + (Y_3 - Y_4 + Y_5)^3 + (Y_3 + Y_4 - Y_5)^3 = 0\,. \end{aligned}$$

We note that $Z_0, \ldots, Z_5$ are projective coordinates, i.e., linearly independent. For that, the only point that requires attention is to verify $a_i + b_i\lambda \neq 0$ for all i. But, as λ is a zero of the auxiliary polynomial, the opposite would imply $a_i + b_i\lambda = 0$ for some $i \in \{0,1,2\}$ and $a_j + b_j\lambda = 0$ for some $j \in \{3,4,5\}$. Then, $\det \left(\begin{smallmatrix} a_i & a_j \\ b_i & b_j \end{smallmatrix}\right) = 0$ and S were singular.

b) is clear. □

4.6. Proposition. —— *Let K be a field and $S = S_{u_0,u_1}^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}$ a non-singular cubic surface in generalized Cayley-Salmon form over K. (Suppose* $\mathrm{char}\, K = 0$.*) For λ a zero of the auxiliary polynomial, put $\mathrm{L}^\lambda := \big\{L_\rho^\lambda \mid \rho\colon \{0,1,2\} \xrightarrow{\cong} \{3,4,5\}\big\}$. Further, for $e := \big\{\binom{012}{345}, \binom{012}{453}, \binom{012}{534}\big\}$ and $o := \big\{\binom{012}{354}, \binom{012}{435}, \binom{012}{543}\big\}$,*

$$\mathrm{L}_e^\lambda := \{L_\rho^\lambda \mid \rho \in e\} \quad \textit{and} \quad \mathrm{L}_o^\lambda := \{L_\rho^\lambda \mid \rho \in o\}\,.$$

i) *Let* $\lambda_1 \neq \lambda_2$ *be zeros of the auxiliary polynomial. Then, the sets* L^{λ_1} *and* L^{λ_2} *are disjoint. In particular, the nine obvious lines* L_{ij} *together with the 18 non-obvious lines* $L_\rho^{\lambda_i}$ *form the set of all the 27 lines on* S.

ii) *Let* $\lambda_1, \lambda_2, \lambda_3$ *be the zeroes of the auxiliary polynomial. Then, the sets* $\mathrm{L}_e^{\lambda_i}$ *and* $\mathrm{L}_o^{\lambda_i}$ *are triplets. In the sense of Proposition 2.8, they form the diagram*

$$\begin{array}{cccc} & \mathrm{L}_e^{\lambda_1} & \mathrm{L}_o^{\lambda_2} & \\ \mathrm{L}_o^{\lambda_3} & & & \mathrm{L}_e^{\lambda_3} \\ & \mathrm{L}_e^{\lambda_2} & \mathrm{L}_o^{\lambda_1} & \end{array}.$$

In particular, the sets of lines $\mathrm{L}_e^{\lambda_1} \cup \mathrm{L}_e^{\lambda_2} \cup \mathrm{L}_e^{\lambda_3}$ *and* $\mathrm{L}_o^{\lambda_1} \cup \mathrm{L}_o^{\lambda_2} \cup \mathrm{L}_o^{\lambda_3}$ *are defined by the pairs of Steiner trihedra, complementary to the one distinguished.*

Proof. i) Assume first that the auxiliary polynomial defines an S_3-extension of K. Then, the lines in L^{λ_1} are defined over $K(\lambda_1)$ and not over K. Analogously, the lines in L^{λ_2} are defined over $K(\lambda_2)$, not over K. As $K(\lambda_1) \cap K(\lambda_2) = K$, this implies the assertion.

Next, suppose that $K = \mathbb{C}$. The family of all non-singular cubic surfaces in Cayley-Salmon form is defined over an open subscheme of

$$\operatorname{Spec} \mathbb{C}[A_0, \dots, A_5, B_0, \dots, B_5] = \mathbf{A}^{12}.$$

The generic fiber is a surface defined over $K = \mathbb{C}(A_0, \dots, A_5, B_0, \dots, B_5)$. It is easy to check that the auxiliary polynomial is irreducible over K and its discriminant is a non-square. Hence, the assertion is true for the generic fiber. Under specialization, intersection numbers are unchanged. In particular, different lines can not specialize to the same. The assertion follows.

For a general base field K, we have that $S^{(a_0,\dots,a_5,b_0,\dots,b_5)}$ is the base change of the corresponding surface over the finitely generated field $K' := \mathbb{Q}(a_0, \dots, a_5, b_0, \dots, b_5)$. As this field injects into $\mathbb{C}$, the proof is complete.

ii) It is readily checked that the sets described are indeed triplets. I.e., that they consist of skew lines. Further, every line in $\mathrm{L}_e^{\lambda_i}$ meets every line in $\mathrm{L}_o^{\lambda_i}$. Thus, these have to be placed in opposite positions.

It remains to be shown that $\mathrm{L}_e^{\lambda_i}$ and $\mathrm{L}_e^{\lambda_j}$ may not be adjacent. For this, again, we may assume that the auxiliary polynomial defines an S_3-extension of K. Suppose $\mathrm{L}_e^{\lambda_1}$ and $\mathrm{L}_e^{\lambda_2}$ were adjacent. Then, together, they would form a sixer. The Galois operation ensures the same for $\mathrm{L}_e^{\lambda_2}$ and $\mathrm{L}_e^{\lambda_3}$ and, as well, for $\mathrm{L}_e^{\lambda_3}$ and $\mathrm{L}_e^{\lambda_1}$. It is, however, impossible that three entries in the diagram are pairwise adjacent. □

5 The norm-trace construction

5.1. —— Let R be a commutative ring with unit and A a commutative R-algebra which is étale and of finite rank. Then, A is, in particular, a locally free R-module.

For an element $a \in A$, we have its norm and trace. In the free case, these are defined by $\mathrm{N}(a) := \det_R(\cdot a\colon A \to A) \in R$ and $\mathrm{tr}(a) := \mathrm{tr}_R(\cdot a\colon A \to A) \in R$. The general case is obtained by gluing.

This definition immediately generalizes to polynomials with coefficients in A. In fact, $A[T_1, \ldots, T_n]$ is étale over $R[T_1, \ldots, T_n]$ of the same rank as A is over R.

5.2. Definition (The norm-trace construction). — Let D be a commutative semisimple algebra of dimension two over $\mathbb{Q}$ and A a commutative D-algebra which is étale of rank three. Further, let $l := c_1T_1 + \ldots + c_4T_4$ be a linear form with coefficients in A and $u \in D$.

Then, we say that the cubic form $\mathrm{NT}_u(l) := \mathrm{tr}(u{\cdot}\mathrm{N}(l))$ is obtained from l and u by the *norm-trace construction*. Correspondingly for the cubic surface $S_u(l)$ over $\mathbb{Q}$ given by $\mathrm{NT}_u(l) = 0$.

5.3. Remarks. — i) Actually, D is either a quadratic number field or isomorphic to $\mathbb{Q} \oplus \mathbb{Q}$. In the latter case, we simply have two étale $\mathbb{Q}$-algebras A_0 and A_1, both of rank three. We start with two linear forms l_0 and l_1 with coefficients in A_0 and A_1, respectively. The norm-trace construction then degenerates to

$$u_0\,\mathrm{N}(l_0) + u_1\,\mathrm{N}(l_1)$$

for $u = (u_0, u_1)$.

ii) An étale algebra of rank three over a field K may be

- $K \times K \times K$,
- the direct product of K with a quadratic field extension, or
- a cubic field extension.

Further, the cubic field extension may be Galois or not. In other words, the corresponding Galois group may be A_3 or S_3. In this language, the degenerate cases correspond to the non-transitive subgroups $\mathbb{Z}/2\mathbb{Z}$ and 0 of S_3.

iii) A is actually always a free D-module. Indeed, D is a semilocal ring. Actually, D is even Artin. Hence, every locally free module of finite rank is free.

iv) As an étale $\mathbb{Q}$-algebra of rank two, D allows two algebra homomorphisms $\iota_0, \iota_1\colon D \to \overline{\mathbb{Q}}$.

On the other hand, as a $\mathbb{Q}$-algebra, A is étale, too. This means, A is a commutative semisimple algebra of rank six. There are exactly six algebra homomorphisms from A to $\overline{\mathbb{Q}}$. Three of them are extensions of ι_0, the others of ι_1. We denote them by τ_0, τ_1, τ_2 and τ_3, τ_4, τ_5, respectively.

In these terms, the norm-trace construction, applied to $l = c_1T_1 + \ldots + c_4T_4$ and u, yields the cubic form $\mathrm{NT}_u(l) = \iota_0(u)l^{\tau_0}l^{\tau_1}l^{\tau_2} + \iota_1(u)l^{\tau_3}l^{\tau_4}l^{\tau_5}$. More explicitly, this is

$$\begin{aligned}&\iota_0(u)[\tau_0(c_1)T_1 + \ldots + \tau_0(c_4)T_4][\tau_1(c_1)T_1 + \ldots + \tau_1(c_4)T_4][\tau_2(c_1)T_1 + \ldots + \tau_2(c_4)T_4]\\ &+ \iota_1(u)[\tau_3(c_1)T_1 + \ldots + \tau_3(c_4)T_4][\tau_4(c_1)T_1 + \ldots + \tau_4(c_4)T_4][\tau_5(c_1)T_1 + \ldots + \tau_5(c_4)T_4]\,.\end{aligned}$$

5.4. Proposition. —— *Suppose, we are given a commutative semisimple $\mathbb{Q}$-algebra D of dimension two and a commutative D-algebra A which is étale of rank three. Further, let $u \in D$ and l be a linear form in four variables $T_1, \dots, T_4$ with coefficients in A.*
Denote by d the dimension of the $\mathbb{Q}$-vector space $\langle l^{\tau_0}, \dots, l^{\tau_5} \rangle \subseteq \Gamma(\mathbf{P}^3, \mathscr{O}(1))$. Fix, finally, a field K such that $K \supseteq \operatorname{im} \tau_0, \dots, \operatorname{im} \tau_5$.
Then,
i) *$l^{\tau_0}, \dots, l^{\tau_5}$ define a rational map*

$$\underline{\iota}\colon S_u(l) \times_{\operatorname{Spec} \mathbb{Q}} \operatorname{Spec} K \dashrightarrow \mathbf{P}^5_K \,.$$

The image of $\underline{\iota}$ is contained in a linear subspace of dimension $d-1$.
ii) *If $d = 4$ then $S_u(l)$ is a cubic surface over $\mathbb{Q}$ such that $S_u(l) \times_{\operatorname{Spec} \mathbb{Q}} \operatorname{Spec} K$ is in generalized Cayley-Salmon form.*
More precisely, if $a_0 l^{\tau_0} + \ldots + a_5 l^{\tau_5} = 0$ and $b_0 l^{\tau_0} + \ldots + b_5 l^{\tau_5} = 0$ are linearly independent relations then $\underline{\iota}$ induces an isomorphism

$$\iota\colon S_u(l) \times_{\operatorname{Spec} \mathbb{Q}} \operatorname{Spec} K \xrightarrow{\cong} S^{(a_0,\dots,a_5,b_0,\dots,b_5)}_{\iota_0(u),\iota_1(u)} \,.$$

iii) *If $d \leq 3$ then $S_u(l)$ is the cone over a, possibly degenerate, cubic curve.*

Proof. i) is standard.
ii) In this case, the forms $l^{\tau_0}, \dots, l^{\tau_5}$ generate the K-vector space $\Gamma(\mathbf{P}^3_K, \mathscr{O}(1))$ of all linear forms. Therefore, they define a closed immersion of $\mathbf{P}^3_K$ into $\mathbf{P}^5_K$. In particular, ι is a closed immersion.

We have the relations $a_0 l^{\tau_0} + \ldots + a_5 l^{\tau_5} = 0$ and $b_0 l^{\tau_0} + \ldots + b_5 l^{\tau_5} = 0$. The cubic surface $S_u(l) \times_{\operatorname{Spec} \mathbb{Q}} \operatorname{Spec} K \subset \mathbf{P}^3_K$ is given by $\iota_0(u) l^{\tau_0} l^{\tau_1} l^{\tau_2} + \iota_1(u) l^{\tau_3} l^{\tau_4} l^{\tau_5} = 0$. Consequently, $\underline{\iota}$ maps $S_u(l) \times_{\operatorname{Spec} \mathbb{Q}} \operatorname{Spec} K$ to the cubic surface in generalized Cayley-Salmon form $S^{(a_0,\dots,a_5,b_0,\dots,b_5)}_{\iota_0(u),\iota_1(u)} \subset \mathbf{P}^5_K$.
iii) is clear. □

5.5. —— As an application of the norm-trace construction, we have an explicit version of Galois descent. For this, some notation has to be fixed.

5.6. Notation. —— For $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, denote by $t_\sigma\colon \operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \overline{\mathbb{Q}}$ the morphism of schemes induced by $\sigma^{-1}\colon \overline{\mathbb{Q}} \leftarrow \overline{\mathbb{Q}}$. This yields a morphism

$$t^{\mathbf{P}^5}_\sigma\colon \mathbf{P}^5_{\overline{\mathbb{Q}}} \longrightarrow \mathbf{P}^5_{\overline{\mathbb{Q}}}$$

of $\overline{\mathbb{Q}}$-schemes which is *twisted by* σ. I.e., compatible with $t_\sigma\colon \operatorname{Spec} \overline{\mathbb{Q}} \to \operatorname{Spec} \overline{\mathbb{Q}}$. Observe that, on $\overline{\mathbb{Q}}$-rational points,

$$t^{\mathbf{P}^5}_\sigma : (x_0 : \ldots : x_5) \mapsto \big(\sigma(x_0) : \ldots : \sigma(x_5)\big) \,.$$

We will usually write t_σ instead of $t_\sigma^{\mathbf{P}^5}$. The morphism t_σ maps the cubic surface $S_{u_0,u_1}^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}$ to $S_{\sigma(u_0),\sigma(u_1)}^{(\sigma(a_0),\ldots,\sigma(a_5),\sigma(b_0),\ldots,\sigma(b_5))}$.

Suppose, for an étale algebra A of rank three over a commutative semisimple $\mathbb{Q}$-algebra D of dimension two and elements $u \in D$ and $a, b \in A$, we have $u_0 = \iota_0(u)$, $u_1 = \iota_1(u)$,

$$a_i = \tau_i(a), \quad \text{and} \quad b_i = \tau_i(b)$$

for $i = 0, \ldots, 5$. Assume that $a_0, \ldots, a_5$ are pairwise different from each other. Then, every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ uniquely determines a permutation $\pi_\sigma \in S_6$ such that

$$\sigma(a_i) = a_{\pi_\sigma(i)} \quad \text{and} \quad \sigma(b_i) = b_{\pi_\sigma(i)}\,.$$

This yields a group homomorphism $\Pi\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to S_6$. We will denote the automorphism of $\mathbf{P}^5$, given by the permutation π on coordinates, by π, too.

Observe that the permutations $\pi_\sigma \in S_6$ preserve the block structure [123][456]. Indeed, σ may eigher interchange the two algebra homomorphisms $\iota_0, \iota_1 : D \to \mathbb{C}$ or not. As a consequence of this, we see that π_σ defines a morphism $S_{\sigma(u_0),\sigma(u_1)}^{(\sigma(a_0),\ldots,\sigma(a_5),\sigma(b_0),\ldots,\sigma(b_5))} \to S_{u_0,u_1}^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}$.

Putting everything together,

$$\pi_\sigma \circ t_\sigma\colon S_{u_0,u_1}^{(a_0,\ldots,a_5,b_0,\ldots,b_5)} \longrightarrow S_{u_0,u_1}^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}$$

is an automorphism twisted by σ. These automorphisms form an operation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $S_{u_0,u_1}^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}$ from the left.

5.7. Theorem (Explicit Galois descent**).** —
Let A be an étale algebra of rank three over a commutative semisimple $\mathbb{Q}$-algebra D of dimension two. For elements $u \in D$ and $a, b \in A$, put $u_0 := \iota_0(u)$, $u_1 := \iota_1(u)$, and, for $i = 0, \ldots, 5$,

$$a_i = \tau_i(a) \quad \text{and} \quad b_i = \tau_i(b)\,.$$

Suppose that a and b are $\mathbb{Q}$-linearly independent. Assume further that $a_0, \ldots, a_5$ are pairwise different from each other.

i) *Then, there exist a cubic surface $S = S^{u_0,u_1}_{(a_0,\ldots,a_5,b_0,\ldots,b_5)}$ over $\mathbb{Q}$ and an isomorphism*

$$\iota\colon S \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\overline{\mathbb{Q}} \xrightarrow{\cong} S_{u_0,u_1}^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}$$

such that, for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the diagram

$$\begin{tikzcd}
S \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\overline{\mathbb{Q}} \arrow[r, "\iota"] & S_{u_0,u_1}^{(a_0,\ldots,a_5,b_0,\ldots,b_5)} \\
S \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\overline{\mathbb{Q}} \arrow[r, "\iota"] \arrow[u, "\mathrm{id}\times t_\sigma"] & S_{u_0,u_1}^{(a_0,\ldots,a_5,b_0,\ldots,b_5)} \arrow[u, "\pi_\sigma\circ t_\sigma"']
\end{tikzcd}$$

commutes.

ii) *The properties given determine S up to a unique isomorphism of $\mathbb{Q}$-schemes.*

iii) *Explicitly, the $\mathbb{Q}$-scheme S may be obtained by the norm-trace construction as follows.*

$$S := S_u(l)$$

for $l = c_1T_1 + \ldots + c_4T_4$ any linear form such that $\mathrm{tr}(al) = 0$, $\mathrm{tr}(bl) = 0$, and $c_0, \ldots, c_3 \in A$ are linearly independent over $\mathbb{Q}$.

Proof. i) and ii) These assertions are particular cases of standard results from the theory of Galois descent [Se, Chapitre V, §4, n° 20, or J, Proposition 2.5]. In fact, the scheme $S^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}_{u_0,u_1}$ is (quasi-)projective over $\overline{\mathbb{Q}}$. Thus, everything which is needed are "descent data", an operation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $S^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}_{u_0,u_1}$ such that $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts by a morphism of $\overline{\mathbb{Q}}$-schemes which is twisted by σ.

iii) The $\mathbb{Q}$-linear system of equations

$$\begin{aligned} \mathrm{tr}(ac) &= 0\,, \\ \mathrm{tr}(bc) &= 0 \end{aligned}$$

has a four-dimensional space $\mathbb{L}$ of solutions. Indeed, the bilinear form $(x, y) \mapsto \mathrm{tr}(xy)$ is non-degenerate [Bou, §8, Proposition 1]. Hence, the first two conditions on l express that $c_0, \ldots, c_3 \in \mathbb{L}$ while the last one is equivalent to saying that $\langle c_0, \ldots, c_3 \rangle$ is a basis of that space.

To exclude the possibility that S degenerates to a cone and to obtain the isomorphism ι, we intend to use Proposition 5.4.ii). This requires to show that the linear forms $l^{\tau_i} = c_1^{\tau_i}T_1 + \ldots + c_4^{\tau_i}T_4$ for $0 \leq i \leq 5$ form a generating system of the vector space of all linear forms. Equivalently, we claim that the 6×4-matrix

$$(c_j^{\tau_i})_{0\leq i\leq 5, 1\leq j\leq 4}$$

is of rank 4.

To prove this, we extend $\{c_1, \ldots, c_4\}$ to a $\mathbb{Q}$-basis $\{c_1, \ldots, c_6\}$ of A. It is enough to verify that the 6×6-matrix $(c_j^{\tau_i})_{0\leq i\leq 5, 1\leq j\leq 6}$ is of full rank. This assertion is actually independent of the particular choice of a basis. We may do the calculations as well with $\{1, a, \ldots, a^5\}$. This yields the Vandermonde matrix

$$\begin{pmatrix} 1 & a^{\tau_0} & \cdots & (a^{\tau_0})^5 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a^{\tau_5} & \cdots & (a^{\tau_1})^5 \end{pmatrix}.$$

of determinant equal to

$$\prod_{i<j} (a^{\tau_i} - a^{\tau_j}) = \prod_{i<j} (a_i - a_j) \neq 0\,.$$

Observe that the six algebra homomorphisms $\tau_i \colon A \to \overline{\mathbb{Q}}$ are given by $a \mapsto a_i$.

Consequently, the linear forms l^{τ_i} yield the desired isomorphism

$$\iota\colon S \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec} K \xrightarrow{\cong} S_{u_0,u_1}^{(a_0,\ldots,a_5,b_0,\ldots,b_5)} .$$

Indeed, we have the equations $\operatorname{tr}(ac_i) = 0$. Explicitly, they express that, for each $i \in \{1, \ldots, 4\}$,

$$0 = (ac_i)^{\tau_0} + \ldots + (ac_i)^{\tau_5} = a_0 c_i^{\tau_0} + \ldots + a_5 c_i^{\tau_5} .$$

This means $a_0 l^{\tau_0} + \ldots + a_5 l^{\tau_5} = 0$. Analogously, we have $b_0 l^{\tau_0} + \ldots + b_5 l^{\tau_5} = 0$.

It remains to verify the commutativity of the diagram. For this, we cover $S_{u_0,u_1}^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}$ by the affine open subsets given by $X_j \neq 0$ for $j = 0, \ldots, 5$. Observe that the morphisms to be compared are both morphisms of $\overline{\mathbb{Q}}$-schemes twisted by σ. Hence, we may compare the pull-back maps between the algebras of regular functions by testing their generators.

For arbitrary $i \neq j$, consider the rational function X_i/X_j. Its pull-back under ι is l^{τ_i}/l^{τ_j}. Therefore, the pull-back of X_i/X_j along the upper left corner is

$$l^{\sigma^{-1}\circ\tau_i}/l^{\sigma^{-1}\circ\tau_j} = (c_1^{\sigma^{-1}\circ\tau_i} t_1 + \ldots + c_4^{\sigma^{-1}\circ\tau_i} t_4)/(c_1^{\sigma^{-1}\circ\tau_j} t_1 + \ldots + c_4^{\sigma^{-1}\circ\tau_j} t_4) .$$

On the other hand, the pull-back of X_i/X_j under $\pi_\sigma \circ t_\sigma$ is $X_{\pi_{\sigma^{-1}}(i)}/X_{\pi_{\sigma^{-1}}(j)}$. Consequently, for the pull-back along the lower right corner, we find

$$\begin{aligned} l^{\tau_{\pi_{\sigma^{-1}}(i)}}/l^{\tau_{\pi_{\sigma^{-1}}(j)}} &= (c_1^{\tau_{\pi_{\sigma^{-1}}(i)}} t_1 + \ldots + c_4^{\tau_{\pi_{\sigma^{-1}}(i)}} t_4)/(c_1^{\tau_{\pi_{\sigma^{-1}}(j)}} t_1 + \ldots + c_4^{\tau_{\pi_{\sigma^{-1}}(j)}} t_4) , \\ &= (c_1^{\sigma^{-1}\circ\tau_i} t_1 + \ldots + c_4^{\sigma^{-1}\circ\tau_i} t_4)/(c_1^{\sigma^{-1}\circ\tau_j} t_1 + \ldots + c_4^{\sigma^{-1}\circ\tau_j} t_4) . \end{aligned}$$

Indeed, the embeddings $\tau_{\pi_{\sigma^{-1}}(i)}, \sigma^{-1}\circ\tau_i\colon A \to \overline{\mathbb{Q}}$ are the same as one may check on the generator T,

$$\tau_{\pi_{\sigma^{-1}}(i)}(T) = a_{\pi_{\sigma^{-1}}(i)} = \sigma^{-1}(a_i) = \sigma^{-1}(\tau_i(T)) = (\sigma^{-1}\circ\tau_i)(T) .$$

This completes the proof. □

5.8. Algorithm (Computation of the Galois descent**).** —— Let two polynomials $g \in \mathbb{Q}[U]$ and $f \in D[V]$ of degrees two and three, respectively, be given which define étale algebras $D := \mathbb{Q}[U]/(g)$ and $A := D[V]/(f)$. Further, let $u \in D$ and $a, b \in A$ be as in Theorem 5.7.

Then, this algorithm computes the Galois descent of the cubic surface $S_{u_0,u_1}^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}$ for $u_0 = \iota_0(u)$, $u_1 = \iota_1(u)$, $a_i = \tau_i(a)$ and $b_i = \tau_i(b)$.

i) Compute, according to the definition, the traces $a_{ij} := \operatorname{tr}(aU^iV^j)$ and $b_{ij} := \operatorname{tr}(bU^iV^j)$ for $i = 0, 1$ and $j = 0, 1, 2$.

ii) Determine the kernel of of the 2×6-matrix

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{10} & a_{11} & a_{12} \\ b_{00} & b_{01} & b_{02} & b_{10} & b_{11} & b_{12} \end{pmatrix}.$$

Choose linearly independent kernel vectors $(c_k^{00}, c_k^{01}, c_k^{02}, c_k^{10}, c_k^{11}, c_k^{12}) \in \mathbb{Q}^6$ for $k = 1, \ldots, 4$.

iii) Compute the norm

$$\mathrm{N}_{A[T_1,\ldots,T_4]/D[T_1,\ldots,T_4]} \left[\sum_{k=1}^{4} \Big(\sum_{\substack{i=0,1 \\ j=0,1,2}} c_k^{ij} U^i V^j \Big) T_k \right].$$

iv) Multiply that cubic form with coefficients in D by u.

v) Finally, apply the trace coefficient-wise and output the resulting cubic form in $T_1, \ldots, T_4$ with 20 rational coefficients.

5.9. Remarks. —— a) To compute the norm of a polynomial F with coefficients in A, we treat $A[T_1, \ldots, T_4]$ as a free $D[T_1, \ldots, T_4]$-module with basis $1, V, V^2$. We establish the 3×3-matrix associated with the multiplication by F map and compute its determinant.

More generally, observe that all the computations in steps i), iii), and iv) are executed in the algebra A which is of dimension six over $\mathbb{Q}$. In order to perform Algorithm 5.8, it is not necessary to realize the Galois hull or any other large algebra on the machine.

b) The case that $D = \mathbb{Q} \oplus \mathbb{Q}$ and $A = \mathbb{Q}[V]/(f_1) \oplus \mathbb{Q}[V]/(f_2)$ is included by taking $g(U) := U^2 - 1$ and

$$f := \tfrac{1+U}{2} f_1 + \tfrac{1-U}{2} f_2 \,.$$

6 The Galois operation on the descent variety

6.1. Lemma. —— *Let A be an étale algebra of rank three over a commutative semisimple $\mathbb{Q}$-algebra D of dimension two and $u \in D$ as well as $a, b \in A$ as in Theorem 5.7.*

a) *If $D \cong \mathbb{Q} \oplus \mathbb{Q}$ then $\Phi_{u_0,u_1}^{(a_0,\ldots,a_5,b_0,\ldots,b_5)} \in \mathbb{Q}[T]$.*

b) *Otherwise, if $D \cong \mathbb{Q}(\sqrt{d})$ then $\sqrt{d}\, \Phi_{u_0,u_1}^{(a_0,\ldots,a_5,b_0,\ldots,b_5)} \in \mathbb{Q}[T]$.*

Proof. We have

$$\begin{aligned} \Phi_{u_0,u_1}^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}(T) = {} & \frac{1}{\iota_0(u)} (\tau_0(a) + \tau_0(b)T)(\tau_1(a) + \tau_1(b)T)(\tau_2(a) + \tau_2(b)T) \\ & - \frac{1}{\iota_1(u)} (\tau_3(a) + \tau_3(b)T)(\tau_4(a) + \tau_4(b)T)(\tau_5(a) + \tau_5(b)T)\,. \end{aligned}$$

The homomorphisms τ_i are acted upon by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. An element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ may either interchange the homomorphisms $\iota_0, \iota_1 \colon D \to \mathbb{C}$ or not. Correspondingly, σ interchanges the two blocks τ_0, τ_1, τ_2 and τ_3, τ_4, τ_5 or not. Anyway, it respects the block structure $[\tau_0\tau_1\tau_2][\tau_3\tau_4\tau_5]$.

a) In this case, no automorphism of $\overline{\mathbb{Q}}$ may interchange ι_0 and ι_1. Hence, $\Phi^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}_{u_0,u_1}$ is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ invariant.

b) Here, the same argument shows that $\Phi^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}$ is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{d}))$ invariant. Hence, $\Phi^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}_{u_0,u_1} \in \mathbb{Q}(\sqrt{d})[T]$. As the polynomial is anti-invariant under the conjugation of $\mathbb{Q}(\sqrt{d})$, the assertion follows. $\square$

6.2. Notation. —— We denote by Γ the subgroup of S_6 of all elements preserving the block structure $[012][345]$. Γ is of order 72. As an abstract group, $\Gamma \cong (S_3 \times S_3) \rtimes \mathbb{Z}/2\mathbb{Z}$.

6.3. Definition. —— Γ operates on the set of all bijections $\rho\colon \{0,1,2\} \to \{3,4,5\}$ as follows. For $\sigma \in \Gamma$, consider the conjugation $\sigma \circ \rho \circ \sigma^{-1}$. If σ interchanges the two blocks then invert this bijection. Otherwise, put $i^\sigma := \sigma \circ \rho \circ \sigma^{-1}$.

6.4. Remarks. —— i) $\rho \mapsto \rho^\sigma$ is an operation of Γ from the left.

ii) If the bijection ρ connects i with j then ρ^σ connects $\sigma(i)$ with $\sigma(j)$.

6.5. Proposition. —— *Let A be an étale algebra of rank three over a commutative semisimple $\mathbb{Q}$-algebra D of dimension two and $u \in D$ as well as $a, b \in A$ as in Theorem 5.7. Further, suppose that $S^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}_{u_0,u_1}$ is non-singular.*

a) *Then,* $\operatorname{disc} \Phi^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}_{u_0,u_1} \in \mathbb{Q}^*$.

b) *Further, on the descent variety $S^{u_0,u_1}_{(a_0,\ldots,a_5,b_0,\ldots,b_5)}$ over $\mathbb{Q}$, there are*

i) *nine obvious lines given by*

$$L_{\{i,j\}}\colon \iota^* X_i = \iota^* X_j = 0$$

for $i = 0, 1, 2$ and $j = 3, 4, 5$,

ii) *18 non-obvious lines given by*

$$L^\lambda_\rho\colon \iota^* Z_0 + \iota^* Z_{\rho(0)} = \iota^* Z_1 + \iota^* Z_{\rho(1)} = 0$$

for λ a zero of the auxiliary polynomial $\Phi^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}_{u_0,u_1}$ and $\rho\colon \{0,1,2\} \to \{3,4,5\}$ a bijection. Here, the coordinates Z_i are given by

$$\begin{aligned} Z_0 &:= -Y_0 + Y_1 + Y_2, \quad Z_1 := Y_0 - Y_1 + Y_2, \quad Z_2 := Y_0 + Y_1 - Y_2, \\ Z_3 &:= -Y_3 + Y_4 + Y_5, \quad Z_4 := Y_3 - Y_4 + Y_5, \quad Z_5 := Y_3 + Y_4 - Y_5 \end{aligned}$$

for

$$Y_i := (a_i + b_i\lambda)X_i, \quad i = 0, \ldots, 5\,.$$

c) *An element* $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *acts on the lines according to the rules*

$$\sigma(L_{\{i,j\}}) = L_{\{\pi_\sigma(i),\pi_\sigma(j)\}}, \qquad \sigma(L^\lambda_\rho) = L^{\lambda^\sigma}_{\rho^{\pi_\sigma}}\,.$$

Proof. a) By Lemma 6.1, the polynomial $\Phi^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}_{u_0,u_1}$ has rational coefficients after multiplication with $\sqrt{d}$ for a suitable $d \in \mathbb{Q}^*$. This implies $\mathrm{disc}\,\Phi^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}_{u_0,u_1} \in \mathbb{Q}$ as the cubic discriminant is homogeneous of degree four in the coefficients. According to Proposition 3.4.ii), smoothness implies $\mathrm{disc}\,\Phi^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}_{u_0,u_1} \neq 0$.
b) The isomorphism

$$\iota\colon S^{u_0,u_1}_{(a_0,\ldots,a_5,b_0,\ldots,b_5)} \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\overline{\mathbb{Q}} \longrightarrow S^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}_{u_0,u_1}$$

is provided by Theorem 5.7. We therefore obtain all the lines by pull-back from $S^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}_{u_0,u_1}$. The formulas for them are given in 4.1 and Proposition 4.5.b). Observe Proposition 4.6 which ensures that the lines given are distinct.
c) From the commutative diagram given in Theorem 5.7.i), we see that the operation of σ on $S^{u_0,u_1}_{(a_0,\ldots,a_5,b_0,\ldots,b_5)} \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\overline{\mathbb{Q}}$ goes over into the automorphism

$$\pi_\sigma \circ t_\sigma\colon S^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}_{u_0,u_1} \longrightarrow S^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}_{u_0,u_1}\,.$$

π_σ permutes the coordinates while t_σ is the operation of σ on the coefficients. □

6.6. Remark. — The nine obvious lines are defined by the pair of Steiner trihedra

$$\big(\{\iota^*E_0, \iota^*E_1, \iota^*E_2\}, \{\iota^*E_3, \iota^*E_4, \iota^*E_5\}\big)$$

which is clearly Galois invariant.

If $D \cong \mathbb{Q} \oplus \mathbb{Q}$ then the Steiner trihedra are Galois invariant themselves. Otherwise, they are permuted by the conjugation of that quadratic number field.

6.7. Corollary. — *Let A be an étale algebra of rank three over a commutative semisimple $\mathbb{Q}$-algebra D of dimension two and $u \in D$ as well as $a, b \in A$ as in Theorem 5.7. Further, suppose that $S^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}_{u_0,u_1}$ is non-singular.*

If, in this situation, the auxiliary polynomial has a rational zero or degenerates to a quadratic polynomial then $S^{u_0,u_1}_{(a_0,\ldots,a_5,b_0,\ldots,b_5)}$ has a Galois invariant double-six.

Proof. First note that $\Phi^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}_{u_0,u_1}$ can not degenerate to a polynomial of degree less than two as we have $\mathrm{disc}(\Phi^{(a_0,\ldots,a_5,b_0,\ldots,b_5)}_{u_0,u_1}) \neq 0$ in the smooth case. If it degenerates to a quadratic polynomial then we have six non-obvious lines corresponding to the zero $\lambda = \infty$.

The assumption therefore implies in any case that there is a Galois invariant set consisting of the nine obvious and six non-obvious lines. The complement of that is a double-six. □

6.8. Corollary. —— *Let $A = D[V]/(f)$ be an étale algebra of rank three over a commutative semisimple $\mathbb{Q}$-algebra D of dimension two and $u \in D$, $a, b \in A$ as in Theorem 5.7. Further, suppose that $S^{(a_0,\dots,a_5,b_0,\dots,b_5)}_{u_0,u_1}$ is non-singular. Let $K \subset \overline{\mathbb{Q}}$ be any subfield.*

i) *Then,* $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ *operates on the 45 tritangent planes by even permutations if and only if*

$$\sqrt{\mathrm{disc}(\Phi^{(a_0,\dots,a_5,b_0,\dots,b_5)}_{u_0,u_1})\ \mathrm{disc}(D)} \in K\,.$$

ii) $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ *does not interchange the two pairs of Steiner trihedra complementary to the distinguished one if and only if*

$$\sqrt{\mathrm{N}_{D/\mathbb{Q}}(\mathrm{disc}\, f)} \in K\,.$$

Proof. i) We will show this result in several steps.

First step. Assume that $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ neither interchanges the zeroes of the auxiliary polynomial nor the embeddings ι_0 and ι_1. Then, the permutation is even.

It will suffice to verify this for the case $\pi_\sigma = (01)$. Then, there are only three invariant lines, namely $L_{\{2,j\}}$ for $j = 3, 4, 5$. It is not hard to check that, for each line L different from these three, the equations defining L and $\sigma(L)$ together form a system of rank three. Hence, we have twelve orbits each consisting of two lines with a point in common.

According to Fact 6.10, the permutation induced on the tritangent planes is a product of 16 two-cycles and, therefore, even.

Second step. Assume that $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ interchanges two zeroes of the auxiliary polynomial but does not interchange ι_0 and ι_1. Then, the permutation is odd.

In view of the first step, it suffices to verify this for the case $\pi_\sigma = \mathrm{id}$. Then, there are 15 invariant lines, the nine obvious ones and the six non-obvious ones corresponding to the invariant root of the auxiliary polynomial.

According to Fact 6.10, the permutation induced on the tritangent planes is a product of 15 two-cycles. Hence, it is odd.

Third step. Assume that $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ stabilizes the zeroes of the auxiliary polynomial but interchanges ι_0 and ι_1. Then, the permutation is odd.

Without restriction, assume that $\pi_\sigma = (03)(14)(25)$. Then, there are 15 invariant lines. These are the three obvious ones $L_{\{1,4\}}$, $L_{\{2,5\}}$, and $L_{\{3,6\}}$ and four for each value of λ. The latter ones correspond to the bijections $\rho\colon \{0, 1, 2\} \to \{3, 4, 5\}$ which fulfill at least one of the three conditions $0 \mapsto 3$, $1 \mapsto 4$, and $2 \mapsto 5$.

Again, according to Fact 6.10, the permutation induced on the tritangent planes is a product of 15 two-cycles and, therefore, odd.

Fourth step. Conclusion.

We see that the operation of σ on the 45 tritangent planes is even if and only if those on $\{\iota_0, \iota_1\}$ and the three zeroes of $\Phi_{u_0,u_1}^{(a_0,\dots,a_5,b_0,\dots,b_5)}$ are of the same parity. This means exactly that

$$\sqrt{\operatorname{disc}(\Phi_{u_0,u_1}^{(a_0,\dots,a_5,b_0,\dots,b_5)})\ \operatorname{disc}(D)}$$

is σ-invariant.

ii) Let $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. According to Proposition 4.6, the two complementary pairs of Steiner trihedra are stable if and only if π_σ does not map the bijections $\{0,1,2\} \to \{3,4,5\}$ from e to o. If π_σ preserves the two blocks $\{0,1,2\}$ and $\{3,4,5\}$ then this means that π_σ must be even. Otherwise, it must be odd. As $A \cong \mathbb{Q}[V]/(\mathrm{N}_{D/\mathbb{Q}}\, f)$, we have exactly that $\sqrt{\operatorname{disc}(\mathrm{N}_{D/\mathbb{Q}}\, f)\ \operatorname{disc}(D)}$ is σ-invariant. We claim that $\operatorname{disc}(\mathrm{N}_{D/\mathbb{Q}}\, f)\ \operatorname{disc}(D)$ equals $\mathrm{N}_{D/\mathbb{Q}}(\operatorname{disc} f)$ up to square factors. Indeed, the definition of the discriminant together with the definition of the resultant [CLO, p. 79], implies, for any two polynomials,

$$\operatorname{disc}(fg) = \operatorname{disc}(f)\operatorname{disc}(g)\operatorname{Res}(f,g)^2.$$

Hence, $\operatorname{disc}(\mathrm{N}_{D/\mathbb{Q}}\, f) = \operatorname{disc}(f)\operatorname{disc}(\overline{f})\operatorname{Res}(f,\overline{f})^2$. But $\operatorname{Res}(f,\overline{f})$ is anti-invariant under the conjugation of D. $\square$

6.9. Remark. — To operate on the 45 tritangent planes by even permutations is a property characterizing the index two subgroup of $W(E_6)$. This is the simple group of order 25 920. On the other hand, the operation on the 27 lines is always even.

6.10. Fact. — *In $W(E_6)$, there are exactly four conjugacy classes of elements of order two. The corresponding operations on the lines and tritangent planes of a non-singular cubic surface are as follows.*

i) *There are* 15 *invariant lines. Then, there are* 15 *invariant tritangent planes. There are* 15 *further orbits which are pairs.*

ii) *There are seven invariant lines. Then, there are five invariant tritangent planes and* 20 *orbits which are formed by pairs.*

iii) *There are three invariant lines. The others form six orbits of two lines which are skew and six orbits of two lines with a point in common.*

In this case, there are seven invariant tritangent planes and 19 *pairs.*

iv) *There are three invariant lines. The others form twelve orbits each consisting of two lines with a point in common.*

Then, there are 13 *invariant tritangent planes. There are* 16 *further orbits which are pairs.*

Proof. This result is essentially due to L. Schläfli [Sch, pp. 114f.] in the guise of his classification of cubic surfaces over the real field. Today, the four conjugacy classes may be seen directly using `GAP`. The operations on the lines and tritangent planes are easily described in the blown-up model. $\square$

7 Examples

7.1. —— Using Algorithm 5.8, we generated a series of examples of smooth cubic surfaces over $\mathbb{Q}$. Our list of examples realizes each of the 63 conjugacy classes of subgroups of $W(E_6)$ which fix a pair of Steiner trihedra but no double-six. It is available at the web page `http:/www.uni-math.gwdg.de/jahnel` of the second author. In this section, we present a few cubic surfaces from our list which are, as we think, of particular interest.

7.2. —— To be more precise, our strategy to generate an example for a particular subgroup $G \subseteq [(S_3 \times S_3) \rtimes \mathbb{Z}/2\mathbb{Z}] \times S_3$ was as follows.

i) If the group stabilizes the two Steiner trihedra then work with $g(U) := U^2 - 1$, i.e. $D = \mathbb{Q} \oplus \mathbb{Q}$. Otherwise, fix a polynomial defining a quadratic number field.

ii) Choose a starting polynomial $f \in D[V]$ such that

- $A := D[V]/(f)$ is étale of rank three over D,
- the Galois group of the degree-six polynomial $\mathrm{N}_{D[V]/Q[V]}(f) \in \mathbb{Q}[V]$ is equal to $\mathrm{pr}_1(G) \subseteq (S_3 \times S_3) \rtimes \mathbb{Z}/2\mathbb{Z}$.

iii) Always put $b := 1$. Further, let $(u, a) \in D \times A$ run through all pairs up to certain height. For each such pair, calculate the auxiliary polynomial Φ. If that has the property desired in order to yield exactly the group G then compute the descent variety in explicit form and terminate immediately.

7.3. Remarks. —— i) The desired property of the auxiliary polynomial might simply be that it be generic with Galois group S_3 or A_3. Several of the groups require more restrictive conditions, for example that $\mathbb{Q}(\sqrt{\operatorname{disc}\Phi})$ coincide with D.

ii) We feel that the actual starting polynomial, we used in the experiment, is somehow irrelevant. Of importance are the algebra A it defines and the element $a \in A$. As in all our examples $\mathbb{Q}[a] = A$, we will present the algebras in the form $A := D[V]/(f)$ for $a = \overline{V}$.

iii) For 42 of the 63 groups, we could start with the algebra $D = \mathbb{Q} \oplus \mathbb{Q}$. In this case, actually $f = (f_0, f_1)$ for $f_0, f_1 \in \mathbb{Q}[V]$. The other 21 groups required a quadratic number field.

7.4. Example. —— Start with $g(U) := U^2 - 1$, i.e. $D := \mathbb{Q} \oplus \mathbb{Q}$, $f_0(V) := V^3 + \frac{1}{2}V + 1$, $f_1(V) := V^3 - 2V^2 + 5$, $u_0 = 1$, and $u_1 = 2$. Both polynomials have Galois group S_3. The resulting non-singular cubic surface S is given by the equation

$$\begin{aligned}18T_1^3 - 40T_1^2T_2 + 37T_1^2T_3 - 30T_1^2T_4 + 68T_1T_2^2 + 4T_1T_2T_3 - 36T_1T_2T_4 - 64T_1T_3^2 \\ - 14T_1T_3T_4 + 38T_1T_4^2 - 24T_2^3 - 6T_2^2T_3 - 12T_2^2T_4 - 72T_2T_3^2 + 64T_2T_3T_4 + 16T_2T_4^2 \\ + 31T_3^3 - 12T_3^2T_4 + 27T_3T_4^2 - 5T_4^3 = 0\,.\end{aligned}$$

Here, the auxiliary polynomial is generic with Galois group S_3. Hence, the Galois group operating on the 27 lines is the maximal $S_3 \times S_3 \times S_3$ stabilizing the two Steiner trihedra of a pair. We have orbit structure $[9, 18]$.

7.5. Example. —— Start with $g(U) := U^2 - 7$, i.e. $D := \mathbb{Q}(\sqrt{7})$, and $f(V) := V^3 + (1-\sqrt{7})V^2 + (-1+\sqrt{7})V + (5-\sqrt{7})$. This is a polynomial with Galois group S_3. Its discriminant is $(810\sqrt{7} - 2376)$, a number of norm 1026^2. Finally, put $u := \sqrt{7}$. The resulting non-singular cubic surface S is given by the equation

$$\begin{aligned}-5T_1^3 + 5T_1^2T_2 + 5T_1^2T_3 + 3T_1T_2^2 - 5T_1T_2T_3 + 5T_1T_2T_4 \\ + 4T_1T_3^2 - T_1T_3T_4 - 6T_1T_4^2 - 6T_2^3 - 3T_2^2T_3 - 6T_2^2T_4 + 2T_2T_3^2 \\ + 2T_2T_3T_4 - 4T_2T_4^2 - 5T_3^3 - 4T_3^2T_4 - 4T_3T_4^2 - 2T_4^3 = 0\,.\end{aligned}$$

The auxiliary polynomial has Galois group S_3.

As $N_{D/\mathbb{Q}}(\mathrm{disc}(f))$ is a perfect square, the Galois group operating on the 27 lines stabilizes three pairs of Steiner trihedra which are complementary in the sense that together they contain all the 27 lines. Actually, it is the maximal group with this property. It is of index two in $[(S_3 \times S_3) \rtimes \mathbb{Z}/2\mathbb{Z}] \times S_3$. We have orbit structure $[9, 9, 9]$.

7.6. Example. —— Start with $g(U) := U^2 - 1$, i.e. $D := \mathbb{Q} \oplus \mathbb{Q}$, $f_0(V) := V^3 + 3V^2 - 9V - 19$, $f_1(V) := V^3 - 15V^2 + \frac{261}{4}V - 85$, $u_0 = 4$, and $u_1 = 1$. Then, Algorithm 5.7 yields the non-singular cubic surface S given by the equation

$$\begin{aligned}9T_1^3 + 4T_1^2T_2 + 6T_1^2T_3 - 3T_1T_2^2 - 2T_1T_2T_3 - 3T_1T_3^2 - T_2^3 - 3T_2^2T_3 \\ - 3T_2^2T_4 - 6T_2T_3^2 + 2T_2T_3T_4 + 11T_2T_4^2 + T_3^3 - 3T_3T_4^2 - T_4^3 = 0\,.\end{aligned}$$

Here, the auxiliary polynomial has Galois group A_3. Even more, its splitting field coincides with that of f_1. Hence, the Galois group operating on the 27 lines is of order three. We have orbit structure $[3, 3, 3, 3, 3, 3, 3, 3, 3]$.

7.7. Remark. —— This group of order three is of particular interest in connection with the Brauer group $\mathrm{Br}(S) = H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S))$ of S. This is an important invariant in the arithmetic of cubic surfaces. For more information on its applications, the reader is referred to Yu. I. Manin's book [Ma].

It turns out that $\mathrm{Br}(S)$ is completely determined by the group operating on the 27 lines. It may take only five [SD, Co] values, 0, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, and $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. A calculation in `GAP` shows that $\mathrm{Br}(S) = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ for exactly one of the 350 conjugacy classes of subgroups in $W(E_6)$.

This group was discussed in [Ma], already, as the group operating on the 27 lines of "$aT_1^3 + T_2^3 + T_3^3 + T_4^3 = 0$" for $a \in \mathbb{Q}(\zeta_3)$ a non-cube. Example 7.6 shows that it appears for cubic surfaces over $\mathbb{Q}$, too.

7.8. Example. —— Start with $g(U) := U^2 - 2$, i.e. $D := \mathbb{Q}(\sqrt{2})$, and $f(V) := V^3 - \frac{3}{2}\sqrt{2}V + \sqrt{2}$. We have $F(V) := \mathrm{N}_{D[V]/Q[V]}(f) = V^6 - \frac{9}{2}V^2 + 6V - 2$, the Galois group of which is $[(S_3 \times S_3) \rtimes \mathbb{Z}/2\mathbb{Z}] \cap A_6$ of order 36. In particular, the Galois group of f itself is A_3.

It turns out that, for $u := (-\sqrt{2}+5)$, the auxiliary polynomial has discriminant, up to square factors, equal to 2. Hence, the Galois operation on the 45 tritangent planes is even. The Galois group G is of order 108. On the 27 lines, the orbit structure is $[9, 18]$.

7.9. Remark. —— This example is of interest from the technical point of view. Observe the following particularities.

The 2-Sylow subgroup of G is cyclic of order four. Actually, G is the maximal even subgroup of $W(E_6)$ stabilizing a pair of Steiner trihedra with this property.

The Galois group of $F(V)$ of order 36 has the same 2-Sylow subgroup, already. It is generated by a permutation of the form (0314)(25). Further, its 3-Sylow subgroup is normal. Hence, we may obtain the cyclic group of order four as a quotient. Consequently, there must be a quadratic extension of $D = \mathbb{Q}(\sqrt{d})$ which is Galois and even cyclic over $\mathbb{Q}$. This causes limitations on D due to the following fact.

7.10. Fact. —— *Let $L/\mathbb{Q}$ be a Galois extension which is cyclic of degree four. Then, the quadratic intermediate field $\mathbb{Q}(\sqrt{d})$ is real and the norm of the fundamental unit is (-1). In particular, $d \equiv 1, 2 \pmod 4$.*

Proof. We have to show that (-1) is a norm from D. For this, according to the Hasse norm theorem, it is sufficient to verify that (-1) is a norm from D_ν for every prime ν. If $[L_w : \mathbb{Q}_p] = 1$ then there is nothing to prove. If $[L_w : \mathbb{Q}_p] = 2$ then the decomposition group is $2\mathbb{Z}/4\mathbb{Z} \subset \mathbb{Z}/4\mathbb{Z}$ which means that $D_\nu = \mathbb{Q}_p$. Thus, there is nothing to prove, either. Finally, if $[L_w : \mathbb{Q}_p] = 4$ then, according to local class field theory, $NL_w^* \subset ND_\nu^* \subset \mathbb{Q}_p^*$, each of index two, such that $\mathbb{Q}_p^*/NL_w^* \cong \mathbb{Z}/4\mathbb{Z}$. The fact that $(-1)^2 = 1$ implies $(-1) \in ND_\nu^*$. □

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