Estimates for Tamagawa numbers of diagonal cubic surfaces[1]

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Abstract

For diagonal cubic surfaces, we give an upper bound for E. Peyre's Tamagawa type number in terms of the coefficients of the defining equation. This bound shows that the reciprocal $\frac{1}{\tau(S)}$ admits a fundamental finiteness property on the set of all diagonal cubic surfaces. As an application, we show that the infinite series of Tamagawa numbers related to the Fano cubic bundles considered by Batyrev and Tschinkel [BT] are indeed convergent.

Key words: Diagonal cubic surface, Diophantine equation, E. Peyre's Tamagawa-type number
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1. Introduction

1.1. —— A conjecture, due to Yu. I. Manin, asserts that the number of $\mathbb{Q}$-rational points of anticanonical height $< B$ on a del Pezzo surface S is asymptotically equal to $\tau B \log^{\operatorname{rk}\operatorname{Pic}(S)-1} B$, for $B \to \infty$. Further, the coefficient $\tau \in \mathbb{R}$ is conjectured to be the Tamagawa-type number $\tau(S)$ introduced by E. Peyre in [Pe]. In the particular case of a cubic surface, the anticanonical height is the same as the naive height.

1.2. ***E. Peyre's constant.*** —— E. Peyre's Tamagawa-type number is defined in [PT, Definition 2.4] as

$$\tau(S) := \alpha(S)\cdot\beta(S) \cdot \lim_{s\to 1}(s-1)^t L(s, \chi_{\operatorname{Pic}(S_{\overline{\mathbb{Q}}})}) \cdot \tau_H(S(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}})$$

for $t = \operatorname{rk}\operatorname{Pic}(S)$.

Here, the factor $\beta(S)$ is simply defined as

$$\beta(S) := \#H^1(\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \operatorname{Pic}(S_{\overline{\mathbb{Q}}})).$$

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$\alpha(S)$ is given as follows [Pe, Définition 2.4]. Let $\Lambda_{\mathrm{eff}}(S) \subset \mathrm{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{R}$ be the cone generated by the effective divisors. Consider the dual cone $\Lambda^{\vee}_{\mathrm{eff}}(S) \subset (\mathrm{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{R})^{\vee}$, defined by

$$\Lambda^{\vee}_{\mathrm{eff}}(S) := \{\mu \in (\mathrm{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{R})^{\vee} \mid \langle \mu, \lambda \rangle \geq 0 \text{ for every } \lambda \in \Lambda_{\mathrm{eff}}(S)\}.$$

Then,

$$\alpha(S) := t \cdot \mathrm{vol}\{\mu \in \Lambda^{\vee}_{\mathrm{eff}}(S) \mid \langle \mu, -K \rangle \leq 1\}.$$

Here, vol denotes the Lebesgue measure on $(\mathrm{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{R})^{\vee}$, normalized such that a primitive cell of the lattice $\mathrm{Pic}(S)^{\vee} \subset (\mathrm{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{R})^{\vee}$ is of measure one.

Further, $L(\,\cdot\,, \chi_{\mathrm{Pic}(S_{\overline{\mathbb{Q}}})})$ is the Artin L-function of the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation $\mathrm{Pic}(S_{\overline{\mathbb{Q}}}) \otimes_{\mathbb{Z}} \mathbb{C}$ which contains the trivial representation t times as a direct summand. Therefore,

$$L(s, \chi_{\mathrm{Pic}(S_{\overline{\mathbb{Q}}})}) = \zeta(s)^t \cdot L(s, \chi_P)$$

and

$$\lim_{s \to 1} (s-1)^t L(s, \chi_{\mathrm{Pic}(S_{\overline{\mathbb{Q}}})}) = L(1, \chi_P)$$

where ζ denotes the Riemann zeta function and P is a representation which does not contain trivial components. [Mu, Corollary 11.5 and Corollary 11.4] show that $L(s, \chi_P)$ has neither a pole nor a zero at $s = 1$. Then, $L(1, \chi_P) > 0$.

Finally, τ_H is the *Tamagawa measure* on the set $S(\mathbb{A}_{\mathbb{Q}})$ of adelic points on S and $S(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} \subseteq S(\mathbb{A}_{\mathbb{Q}})$ consists of those adelic points which are orthogonal to the Brauer group $\mathrm{Br}(S)$ with respect to the Brauer-Manin pairing

$$S(\mathbb{A}_{\mathbb{Q}}) \times \mathrm{Br}(S) \to \mathbb{Q}/\mathbb{Z}, \qquad (\{x_\nu\}, \alpha) \mapsto \sum_{\nu} \mathrm{inv}_\nu\, \alpha|_{x_\nu}.$$

1.3. —— As S is projective, we have

$$S(\mathbb{A}_{\mathbb{Q}}) = \prod_{\nu \in \mathrm{Val}(\mathbb{Q})} S(\mathbb{Q}_\nu).$$

τ_H is defined to be a product measure $\tau_H := \prod_{\nu \in \mathrm{Val}(\mathbb{Q})} \tau_\nu$.

For a prime number p, the local measure τ_p is given as follows. Let $a \in S(\mathbb{Z}/p^k\mathbb{Z})$ and put $\mathfrak{U}^{(k)}_a := \{x \in S(\mathbb{Q}_p) \mid x \equiv a \pmod{p^k}\}$. Then,

$$\tau_p(\mathfrak{U}^{(k)}_a) := \det(1 - p^{-1}\,\mathrm{Frob}_p \mid \mathrm{Pic}(S_{\overline{\mathbb{Q}}})^{I_p}) \cdot \lim_{m \to \infty} \frac{\#\{y \in S(\mathbb{Z}/p^m\mathbb{Z}) \mid y \equiv a \pmod{p^k}\}}{p^{m \dim S}}.$$

Here, $\mathrm{Pic}(S_{\overline{\mathbb{Q}}})^{I_p}$ denotes the fixed module under the inertia group.

The measure τ_∞ is described in [Pe, Lemme 5.4.7]. In the case of a hypersurface of degree d in $\mathbf{P}^n$, defined by the equation $f = 0$, this yields

$$\tau_\infty(U) = \frac{n+1-d}{2} \int\limits_{\substack{CU \\ |x_0|, \ldots, |x_n| \leq 1}} \omega_{\mathrm{Leray}}$$

for every Borel set $U \subset S(\mathbb{R})$. Here, ω_{Leray} is the *Leray measure* on the cone $CS(\mathbb{R}) \subset \mathbb{R}^{n+1}$ associated with the equation $f = 0$. It is given by the differential form $\frac{1}{|\partial f/\partial x_0|}\, dx_1 \wedge \ldots \wedge dx_n$.

1.4. Remark. — There is a "(hyper)surface area" ω_{hyp} typically introduced for hypersurfaces in $\mathbb{R}^{n+1}$ in multivariable calculus. That measure is actually the canonical volume associated with the Riemannian metric $CS(\mathbb{R})$ inherits from $\mathbb{R}^{n+1}$ [Di, 20.8.6.2]. The Leray measure is related to the hypersurface area by the formula $\omega_{\text{Leray}} = \frac{1}{\|\operatorname{grad} f\|}\,\omega_{\text{hyp}}$.

1.5. *The main result.* — At least for diagonal cubic surfaces, the reciprocal $\frac{1}{\tau(S)}$ admits a fundamental finiteness property. More precisely, we will prove the following result.

Theorem. *For $\mathfrak{a} = (a_0, \ldots, a_3) \in (\mathbb{Z}\backslash\{0\})^4$ any vector, we denote by $S^{\mathfrak{a}}$ the cubic surface in $\mathbf{P}^3_{\mathbb{Q}}$ given by $a_0x_0^3 + \ldots + a_3x_3^3 = 0$. Then, for each $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that*

$$\frac{1}{\tau(S^{\mathfrak{a}})} \geq C(\varepsilon) \cdot \mathrm{H}_{\text{naive}}\Big(\frac{1}{a_0} : \ldots : \frac{1}{a_3}\Big)^{\frac{1}{3}-\varepsilon}.$$

1.6. Corollary (Fundamental finiteness**).** — *For each $T > 0$, there are only finitely many diagonal cubic surfaces $S^{\mathfrak{a}}\colon a_0x_0^3 + \ldots + a_3x_3^3 = 0$ in $\mathbf{P}^3_{\mathbb{Q}}$ such that $\tau(S^{\mathfrak{a}}) > T$.*

1.7. Remark. — For diagonal quartic threefolds, these results were shown in [EJ]. The case of the classical cubic surfaces is, however, more complicated.

The reason for this is that quartic threefolds are of geometric Picard rank one. Hence, the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation considered was always trivial and the L-factor was automatically equal to 1. In the situation of a diagonal cubic surface, the factors $\lim\limits_{s\to 1}(s-1)^t L(s, \chi_{\operatorname{Pic}(S_{\overline{\mathbb{Q}}})})$ add new difficulty.

There is also a difference concerning the factors α and β. This point is, however, of minor significance. For quartic threefolds, we always had $\alpha(S) = \beta(S) = 1$. For cubic surfaces, these factors may vary but it is not at all hard to estimate them.

1.8. *An application.* — For Fano varieties of dimension ≥ 3, the obvious generalization of Manin's conjecture is known to be wrong. Due to Batyrev and Tschinkel [BT], there are counterexamples of Picard rank 2. These are smooth hypersurfaces $X \subset \mathbf{P}^n \times \mathbf{P}^3$ of bidegree $(1, 3)$. Such a hypersurface is equipped with a fibration into cubic surfaces given by the projection to the first factor. It is assumed that those are diagonal.

Seemingly, many people believe that the actual growth of the number of $\mathbb{Q}$-rational points on X is dominated by the fibres of Picard rank 4. This means, the asymptotics is expected to be $\tau B \log^3 B$ for

$$\tau := \sum_{\substack{x\in\mathbf{P}^n(\mathbb{Q})\\ S^{\iota(x)} \text{ non-singular}\\ \operatorname{rk}\operatorname{Pic}(S^{\iota(x)})=4}} \frac{1}{\mathrm{H}^n_{\text{naive}}(x)}\,\tau(S^{\iota(x)})\,. \tag{1}$$

Here, $\iota\colon \mathbf{P}^n \dashrightarrow (\mathbf{P}^3)^\vee$ is the linear map defined by the fibration.

As an application of Theorem 1.5, we will show that the series (1) are indeed convergent. For this, as will turn out, it is already sufficient that the Tamagawa numbers of diagonal cubic surfaces are uniformly bounded. Details will be given in section 3.

2. Estimates for Peyre's constant

Consider a general diagonal cubic surface $S^{(a_0,\dots,a_3)} \subset \mathbf{P}^3_{\mathbb{Q}}$ given by

$$a_0x_0^3 + \ \dots \ + a_3x_3^3 = 0\,.$$

Our goal is to establish the estimate for $\tau^{(a_0,\dots,a_3)} := \tau(S^{(a_0,\dots,a_3)})$ formulated in Theorem 1.5. For this, in the subsections below, we will give an individual estimate for each of the factors occurring in the definition of $\tau(S^{(a_0,\dots,a_3)})$.

2.1. Estimates for α and β

2.1.1. — Recall that on a smooth cubic surface $\mathscr{S}$ over an algebraically closed field, there are exactly 27 lines. For the Picard group, which is isomorphic to $\mathbb{Z}^7$, the classes of these lines form a system of generators.

2.1.2. Notation. — i) The set $\mathscr{L}$ of the 27 lines is equipped with the intersection product $\langle\ ,\ \rangle\colon \mathscr{L}{\times}\mathscr{L} \to \{-1, 0, 1\}$. The pair $(\mathscr{L}, \langle\ ,\ \rangle)$ is the same for all smooth cubic surfaces. It is well known [Ma, Theorem 23.9.ii] that the group of permutations of $\mathscr{L}$ respecting $\langle\ ,\ \rangle$ is isomorphic to $W(E_6)$. We fix such an isomorphism.

Denote by $F \subset \mathrm{Div}(\mathscr{S})$ the group generated by the 27 lines and by $F_0 \subset F$ the subgroup of principal divisors. Then, F is equipped with an operation of $W(E_6)$ such that F_0 is a $W(E_6)$-submodule. We have $\mathrm{Pic}(\mathscr{S}) \cong F/F_0$.

ii) If S is a smooth cubic surface over $\mathbb{Q}$ then $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ operates canonically on the set $\mathscr{L}_S$ of the 27 lines on $S_{\overline{\mathbb{Q}}}$. Fix a bijection $i_S\colon \mathscr{L}_S \stackrel{\cong}{\longrightarrow} \mathscr{L}$ respecting the intersection pairing. This induces a group homomorphism $\iota_S\colon\ \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to W(E_6)$. We denote its image by $G \subset W(E_6)$.

2.1.3. Lemma. — *There is a constant c such that, for all smooth cubic surfaces S over $\mathbb{Q}$,*

$$1 \leq \beta(S) \leq c\,.$$

Proof. By definition, $\beta(S) = \#H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}}))$. Using the notation just introduced, we may write $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) = H^1(G, F/F_0)$.

Note that this cohomology group is always finite. Indeed, since G is a finite group and F/F_0 is a finite $\mathbb{Z}[G]$-module, the description via the standard complex shows it is finitely generated. Further, it is annihilated by $\#G$.

$H^1(G, F/F_0)$ depends only on the subgroup $G \subset W(E_6)$ occurring. For that, there are finitely many possibilities. This implies the claim. □

2.1.4. Remarks. — i) A more precise consideration [Ma, Proposition 31.3] yields a canonical isomorphism

$$H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(S_{\overline{\mathbb{Q}}})) \cong \mathrm{Hom}\,((NF \cap F_0)/NF_0, \mathbb{Q}/\mathbb{Z})\,.$$

Here, N is the norm map under the operation of G.

As an application of this, one may inspect the 350 conjugacy classes of subgroups of $W(E_6)$ using `GAP`. The calculations show that the lemma is actually true for $c = 9$.

ii) Diagonal cubic surfaces actually provide only 16 of the 350 conjugacy classes. Eight of them may be realized over $\mathbb{Q}$, the others over $\mathbb{Q}(\zeta_3)$ [CTKS].

2.1.5. Lemma. —— *There are positive constants c_1 and c_2 such that, for all smooth cubic surfaces S over $\mathbb{Q}$ satisfying $S(\mathbb{A}_\mathbb{Q}) \neq \emptyset$,*

$$c_1 \leq \alpha(S) \leq c_2.$$

Proof. Again, we claim that $\alpha(S)$ is completely determined by the group $G \subset W(E_6)$. Thus, suppose that we do not have the full information available about what surface S is but are given the group G only.

The assumption $S(\mathbb{A}_\mathbb{Q}) \neq \emptyset$ makes sure that $\mathrm{Pic}(S) \cong \mathrm{Pic}(S_{\overline{\mathbb{Q}}})^G$ [KT, Remark 3.2.ii)]. We may therefore write $\mathrm{Pic}(S) \cong (F/F_0)^G$. The effective cone

$$\Lambda_{\mathrm{eff}}(S) \subset \mathrm{Pic}(S) \otimes_\mathbb{Z} \mathbb{C} \cong (F/F_0)^G \otimes_\mathbb{Z} \mathbb{C}$$

is generated by the symmetrizations of the classes $\ell_1, \ldots, \ell_{27}$ of the 27 lines in F. In particular, it is determined by G, completely. Further, we have $K = -\frac{1}{9}(\ell_1 + \ldots + \ell_{27})$. These data are sufficient to compute $\alpha(S)$ according to its very definition. $\square$

2.1.6. Remark. —— Here, we do not know the optimal values of c_1 and c_2 in explicit form. $\alpha(S)$ has not yet been computed in all cases.

*2.2. An estimate for the **L**-factor*

2.2.1. —— In the case of the diagonal cubic surface $S^{(a_0,\ldots,a_3)} \subset \mathbf{P}^3_\mathbb{Q}$, given by $a_0x_0^3 + \ldots + a_3x_3^3 = 0$ for $a_0, \ldots, a_3 \in \mathbb{Z} \setminus \{0\}$, the 27 lines on $S^{(a_0,\ldots,a_3)}$ may easily be written down explicitly. Indeed, for each pair $(i, j) \in (\mathbb{Z}/3\mathbb{Z})^2$, the system

$$\begin{aligned} \sqrt[3]{a_0}\, x_0 + \zeta_3^i \sqrt[3]{a_1}\, x_1 &= 0 \\ \sqrt[3]{a_2}\, x_2 + \zeta_3^j \sqrt[3]{a_3}\, x_3 &= 0 \end{aligned}$$

of equations defines a line on $S^{(a_0,\ldots,a_3)}$. Decomposing the index set $\{0, \ldots, 3\}$ differently into two subsets of two elements each yields all the lines. In particular, we see that the 27 lines may be defined over $K = \mathbb{Q}(\zeta_3, \sqrt[3]{a_1/a_0}, \sqrt[3]{a_2/a_0}, \sqrt[3]{a_3/a_0})$.

2.2.2. —— This is an abelian extension of $\mathbb{Q}(\zeta_3)$. Therefore, the irreducible representations of $\mathrm{Gal}(K/\mathbb{Q})$ are at most two-dimensional. Besides the trivial representation, there is the non-trivial Dirichlet character λ of $\mathbb{Q}(\zeta_3)/\mathbb{Q}$. The two-dimensional irreducible representations are actually representations of a factor group of the form $\mathrm{Gal}(\mathbb{Q}(\zeta_3, \sqrt[3]{a_0^{e_0} \cdot \ldots \cdot a_3^{e_3}})/\mathbb{Q}) \cong S_3$ for $e_0, \ldots, e_3 \in \{0, 1, 2\}$.

2.2.3. Lemma. —— *Let a and b be integers different from zero. Then,*

$$\left|\mathrm{Disc}\,(\mathbb{Q}(\zeta_3, \sqrt[3]{ab^2})/\mathbb{Q})\right| \leq 3^9 a^4 b^4.$$

Proof. We have, at first,

$$\begin{aligned} \left|\mathrm{Disc}\,(\mathbb{Q}(\zeta_3, \sqrt[3]{ab^2})/\mathbb{Q})\right| &\leq \left|\mathrm{Disc}\,(\mathbb{Q}(\zeta_3)/\mathbb{Q})\right|^3 \cdot \mathrm{Disc}\,(\mathbb{Q}(\sqrt[3]{ab^2})/\mathbb{Q})^2 \\ &= 27 \cdot \mathrm{Disc}\,(\mathbb{Q}(\sqrt[3]{ab^2})/\mathbb{Q})^2. \end{aligned}$$

Further, by [De, §4], we know

$$\left|\mathrm{Disc}\,(\mathbb{Q}(\sqrt[3]{ab^2})/\mathbb{Q})\right| \leq 3^3 a^2 b^2.$$

This shows $\left|\mathrm{Disc}\,(\mathbb{Q}(\zeta_3, \sqrt[3]{ab^2})/\mathbb{Q})\right| \leq 3^9 a^4 b^4$. $\square$

2.2.4. Proposition. —— *For each $\varepsilon > 0$, there exist positive constants c_1 and c_2 such that*

$$c_1 \cdot |a_0 \cdot \ldots \cdot a_3|^{-\varepsilon} < \lim_{s \to 1} (s-1)^t L\,(s, \chi_{\mathrm{Pic}(S^{(a_0,\ldots,a_3)}_{\overline{\mathbb{Q}}})}) < c_2 \cdot |a_0 \cdot \ldots \cdot a_3|^{\varepsilon}$$

for all $(a_0, \ldots, a_3) \in (\mathbb{Z} \backslash \{0\})^4$. Here, $t = \mathrm{rk}\,\mathrm{Pic}(S)$.

Proof. The Galois representation $\mathrm{Pic}(S^{(a_0,\ldots,a_3)}_{\overline{\mathbb{Q}}}) \otimes_{\mathbb{Z}} \mathbb{C}$ contains the trivial representation t times as a direct summand. Therefore,

$$L(s, \chi_{\mathrm{Pic}(S^{(a_0,\ldots,a_3)}_{\overline{\mathbb{Q}}})}) = \zeta(s)^t \cdot L\,(s, \chi_P)$$

where ζ denotes the Riemann zeta function and P is a representation which does not contain trivial components. All we need to show is

$$c_1 \cdot |a_0 \cdot \ldots \cdot a_3|^{-\varepsilon} < L\,(1, \chi_P) < c_2 \cdot |a_0 \cdot \ldots \cdot a_3|^{\varepsilon}.$$

$L\,(\,\cdot\,, \chi_P)$ is the product [Ne, Chapter VII, Theorem (10.4).ii)] of not more than six factors of the form $L\,(\,\cdot\,, \lambda)$ for λ the non-trivial Dirichlet character of $\mathbb{Q}(\zeta_3)/\mathbb{Q}$ and at most three factors which are Artin-L-functions $L\,(\,\cdot\,, \nu^K)$ for two-dimensional irreducible representations.

Here, $K = \mathbb{Q}(\zeta_3, \sqrt[3]{a_0^{e_0} \cdot \ldots \cdot a_3^{e_3}})$ for certain $e_0, \ldots, e_3 \in \{0, 1, 2\}$. As $L\,(1, \lambda)$ does not depend on $a_0, \ldots, a_3$, at all, it will suffice to show

$$c_1(\varepsilon) \cdot |a_0 \cdot \ldots \cdot a_3|^{-\varepsilon} < L\,(1, \nu^K) < c_2(\varepsilon) \cdot |a_0 \cdot \ldots \cdot a_3|^{\varepsilon}$$

for each $\varepsilon > 0$.

ν^K is the only irreducible two-dimensional character of $\mathrm{Gal}(K/\mathbb{Q}) \cong S_3$. For that reason, by virtue of [Ne, Chapter VII, Corollary (10.5)], we have

$$\begin{aligned} \zeta_K(s) &= \zeta_{\mathbb{Q}}(s) \cdot L\,(s, \lambda) \cdot L\,(s, \nu^K)^2 \\ &= \zeta_{\mathbb{Q}(\zeta_3)}(s) \cdot L\,(s, \nu^K)^2 \end{aligned}$$

for a complex variable s. It, therefore, suffices in our particular situation to estimate the residue $\mathrm{res}_{s=1}\, \zeta_K(s)$ of the Dedekind zeta function of K.

An estimate from above has been given by C. L. Siegel. In view of the analytic class number formula, his [Si, Satz 1] gives

$$\begin{aligned} \mathop{\mathrm{res}}_{s=1} \zeta_K(s) &< C\,[\log \mathrm{Disc}(K/\mathbb{Q})]^5 \\ &\leq C\,[\log(3^9 a_0^4 a_1^4 a_2^4 a_3^4)]^5 \\ &= C\,[4 \log |a_0 \cdot \ldots \cdot a_3| + 9 \log 3]^5 \end{aligned}$$

for a certain constant C. The final term is less than $c_2(\varepsilon) \cdot |a_0 \cdot \ldots \cdot a_3|^{\varepsilon}$ for every $\varepsilon > 0$.

On the other hand, H. M. Stark [St, formula (1)] shows

$$\mathop{\mathrm{res}}_{s=1} \zeta_K(s) > C(\varepsilon) \cdot \mathrm{Disc}(K/\mathbb{Q})^{-\varepsilon/4}$$

for every $\varepsilon > 0$ which implies $\mathop{\mathrm{res}}_{s=1} \zeta_K(s) > c_1(\varepsilon) \cdot |a_0 \cdot \ldots \cdot a_3|^{-\varepsilon}$. □

2.3. An estimate for the factors at the finite places

2.3.1. Lemma. —– *There are two positive constants* c_1 *and* c_2 *such that, for all* $a_0, \ldots, a_3 \in \mathbb{Z}\backslash\{0\}$,

$$c_1 < \prod_{\substack{p \text{ prime} \\ p \nmid 3a_0 \cdot \ldots \cdot a_3}} \tau_p(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)) < c_2 \, .$$

Proof. For a prime p of good reduction, Hensel's Lemma implies

$$\tau_p(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)) = \det(1 - p^{-1} \operatorname{Frob}_p \mid \operatorname{Pic}(S_{\overline{\mathbb{Q}}})) \cdot \frac{\# S^{(a_0,\ldots,a_3)}(\mathbb{F}_p)}{p^2}.$$

Further, for the number of points on a non-singular cubic surface over a finite field, the Lefschetz trace formula can be made completely explicit [Ma, Theorem 27.1]. It shows

$$\# S^{(a_0,\ldots,a_3)}(\mathbb{F}_p) = p^2 + p \cdot \operatorname{tr}(\operatorname{Frob}_p \mid \operatorname{Pic}(S_{\overline{\mathbb{Q}}})) + 1 \, .$$

Denoting the eigenvalues of the Frobenius on $\operatorname{Pic}(S_{\overline{\mathbb{Q}}})$ by $\lambda_1, \ldots, \lambda_7$, we find

$$\begin{aligned} \tau_p(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)) &= (1 - \lambda_1 p^{-1})(1 - \lambda_2 p^{-1}) \cdot \ \ldots \ \cdot (1 - \lambda_7 p^{-1}) \\ &\qquad\qquad\qquad\qquad \cdot [1 + (\lambda_1 + \cdots + \lambda_7) p^{-1} + p^{-2}] \\ &= (1 - \sigma_1 p^{-1} + \sigma_2 p^{-2} - \sigma_3 p^{-3} + \ldots - \sigma_7 p^{-7})(1 + \sigma_1 p^{-1} + p^{-2}) \\ &= 1 + (1 - \sigma_1^2 + \sigma_2) p^{-2} - (\sigma_1 - \sigma_1\sigma_2 + \sigma_3) p^{-3} + \ \ldots \\ &\qquad \ldots - (\sigma_5 - \sigma_1\sigma_6 + \sigma_7) p^{-7} + (\sigma_6 - \sigma_1\sigma_7) p^{-8} - \sigma_7 p^{-9} \end{aligned}$$

where σ_i denote the elementary symmetric functions in $\lambda_1, \ldots, \lambda_7$.

We know $|\lambda_i| = 1$ for all i. Estimating very roughly, we have $|\sigma_j| \leq \binom{7}{j} \leq 7^j$ and see

$$\begin{aligned} 1 - 99p^{-2} - 7 \cdot 99 p^{-3} - \ldots - 7^7 \cdot 99 p^{-9} \leq \tau_p(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)) \leq \\ \leq 1 + 99p^{-2} + 7 \cdot 99 p^{-3} + \ldots + 7^7 \cdot 99 p^{-9} \, . \end{aligned}$$

I.e., $1 - 99p^{-2} \frac{1}{1-7/p} < \tau_p(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)) < 1 + 99p^{-2} \frac{1}{1-7/p}$. The infinite product over all $1 - 99p^{-2} \frac{1}{1-7/p}$ (respectively $1 + 99p^{-2} \frac{1}{1-7/p}$) is convergent.

The left hand side is positive for $p > 13$. For the small primes remaining, we need a better lower bound. For this, note that a cubic surface over a finite field $\mathbb{F}_p$ always has at least one $\mathbb{F}_p$-rational point. This yields $\tau_p(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)) \geq (1 - 1/p)^7/p^2 > 0$. □

2.3.2. Remark. —– It will require by far more labour to estimate the product over the finitely many bad primes, uniformly over all diagonal cubic surfaces.

2.3.3. Notation. —– i) For a prime number p and an integer $x \neq 0$, we put $x^{(p)} := p^{\nu_p(x)}$. Note $x^{(p)} = 1/\|x\|_p$ for the normalized p-adic valuation.

ii) For integers $x_1, \ldots, x_n$, not all equal to zero, we write

$$\gcd_p(x_1, \ldots, x_n) := [\gcd(x_1, \ldots, x_n)]^{(p)}.$$

Observe, if $x_1, \ldots, x_n \neq 0$ then we have $\gcd_p(x_1, \ldots, x_n) = \gcd(x_1^{(p)}, \ldots, x_n^{(p)})$.

iii) By putting $\nu(x) := \min\limits_{\substack{\xi \in \mathbb{Z}_p \\ x = (\xi \bmod p^r)}} \nu(\xi)$, we carry the *p-adic valuation* from $\mathbb{Z}_p$ over to $\mathbb{Z}/p^r\mathbb{Z}$.

Note that any $0 \neq x \in \mathbb{Z}/p^r\mathbb{Z}$ has the form $x = \varepsilon \cdot p^{\nu(x)}$ where $\varepsilon \in (\mathbb{Z}/p^r\mathbb{Z})^*$ is a unit. Clearly, ε is unique only in the case $\nu(x) = 0$.

2.3.4. Definition. —— For $(a_0, \ldots, a_3) \in \mathbb{Z}^4$, $r \in \mathbb{N}$, and $\nu_0, \ldots, \nu_3 \leq r$, put

$$N^{(r)}_{\nu_0,\ldots,\nu_3;a_0,\ldots,a_3} := \{ (x_0, \ldots, x_3) \in (\mathbb{Z}/p^r\mathbb{Z})^4 \mid \nu(x_0) = \nu_0, \ldots, \nu(x_3) = \nu_3;\ a_0x_0^3 + \ldots + a_3x_3^3 = 0 \in \mathbb{Z}/p^r\mathbb{Z} \}.$$

For the particular case $\nu_0 = \ldots = \nu_3 = 0$, we will write $Z^{(r)}_{a_0,\ldots,a_3} := N^{(r)}_{0,\ldots,0;a_0,\ldots,a_3}$, i.e.,

$$Z^{(r)}_{a_0,\ldots,a_3} = \{ (x_0, \ldots, x_3) \in [(\mathbb{Z}/p^r\mathbb{Z})^*]^4 \mid a_0x_0^3 + \ldots + a_3x_3^3 = 0 \in \mathbb{Z}/p^r\mathbb{Z} \}.$$

We will use the notation $z^{(r)}_{a_0,\ldots,a_3} := \#Z^{(r)}_{a_0,\ldots,a_3}$.

2.3.5. Sublemma. —— *If* $p^k | a_0, \ldots, a_3$ *and* $r > k$ *then we have*

$$z^{(r)}_{a_0,\ldots,a_3} = p^{4k} \cdot z^{(r-k)}_{a_0/p^k,\ldots,a_3/p^k}.$$

Proof. Since $a_0x_0^3 + \ldots + a_3x_3^3 = p^k(a_0/p^k \cdot x_0^3 + \ldots + a_3/p^k \cdot x_3^3)$, there is a surjection

$$\iota\colon Z^{(r)}_{a_0,\ldots,a_3} \longrightarrow Z^{(r-k)}_{a_0/p^k,\ldots,a_3/p^k},$$

given by $(x_0, \ldots, x_3) \mapsto ((x_0 \bmod p^{r-k}), \ldots, (x_3 \bmod p^{r-k}))$. The kernel of the homomorphism of modules underlying ι is $(p^{r-k}\mathbb{Z}/p^r\mathbb{Z})^4$. □

2.3.6. Lemma. —— *Assume* $\gcd_p(a_0, \ldots, a_4) = p^k$. *Then, there is an estimate*

$$z^{(r)}_{a_0,\ldots,a_4} \leq 3p^{3r+k}.$$

Proof. Suppose first that $k = 0$. This means, one of the coefficients is prime to p. Without restriction, assume $p \nmid a_0$.

For any $(x_1, x_2, x_3) \in (\mathbb{Z}/p^r\mathbb{Z})^3$, there appears an equation of the form $a_0x_0^3 = c$. It cannot have more than three solutions in $(\mathbb{Z}/p^r\mathbb{Z})^*$. Indeed, for p odd, this follows directly from the fact that $(\mathbb{Z}/p^r\mathbb{Z})^*$ is a cyclic group. On the other hand, in the case $p = 2$, we have $(\mathbb{Z}/2^r\mathbb{Z})^* \cong \mathbb{Z}/2^{r-2}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Again, there are only up to three solutions possible.

The general case may now easily be deduced from Sublemma 2.3.5. Indeed, if $k < r$ then

$$z^{(r)}_{a_0,\ldots,a_3} = p^{4k} \cdot z^{(r-k)}_{a_0/p^k,\ldots,a_3/p^k} \leq p^{4k} \cdot 3p^{3(r-k)} = 3p^{3r+k}.$$

On the other hand, if $k \geq r$ then the assertion is completely trivial since

$$z^{(r)}_{a_0,\ldots,a_3} = \#Z^{(r)}_{a_0,\ldots,a_3} < p^{4r} \leq p^{3r+k} < 3p^{3r+k}.$$ □

2.3.7. Remark. —— The proof shows that in the case $p \equiv 2 \pmod 3$ one could reduce the coefficient to 1. Unfortunately, this observation does not lead to a substantial improvement of our final result.

2.3.8. Lemma. — *Let $r \in \mathbb{N}$ and $\nu_0, \ldots, \nu_3 \leq r$. Then,*

$$\#N^{(r)}_{\nu_0,\ldots,\nu_3;a_0,\ldots,a_3} = \frac{z^{(r)}_{p^{3\nu_0}a_0,\,\ldots\,,p^{3\nu_3}a_3} \cdot \varphi(p^{r-\nu_0}) \cdot \ldots \cdot \varphi(p^{r-\nu_3})}{\varphi(p^r)^4}\,.$$

Proof. As $p^{3\nu_0}a_0x_0^3 + \ldots + p^{3\nu_3}a_3x_3^3 = a_0(p^{\nu_0}x_0)^3 + \ldots + a_3(p^{\nu_3}x_3)^3$, we have a surjection

$$\pi\colon Z^{(r)}_{p^{3\nu_0}a_0,\ldots,p^{3\nu_3}a_3} \longrightarrow N^{(r)}_{\nu_0,\ldots,\nu_3;a_0,\ldots,a_3}\,,$$

given by $(x_0, \ldots, x_3) \mapsto (p^{\nu_0}x_0, \ldots, p^{\nu_3}x_3)$.

For $i = 0, \ldots, 3$, consider the mapping $\iota\colon \mathbb{Z}/p^r\mathbb{Z} \to \mathbb{Z}/p^r\mathbb{Z}$, $x \mapsto p^{\nu_i}x$. If $\nu_i = r$ then ι is the zero map. All $\varphi(p^r) = (p-1)p^{r-1}$ units are mapped to zero. Otherwise, observe that ι is $p^{\nu_i} : 1$ onto its image. Further, $\nu(\iota(x)) = \nu_i$ if and only if x is a unit. By consequence, π is $(K^{(\nu_0)} \cdot \ldots \cdot K^{(\nu_3)}) : 1$ when we put $K^{(\nu)} := p^\nu$ for $\nu < r$ and $K^{(r)} := (p-1)p^{r-1}$. Summarizing, we could have written $K^{(\nu)} := \varphi(p^r)/\varphi(p^{r-\nu})$. The assertion follows. $\square$

2.3.9. Corollary. — *Let $(a_0, \ldots, a_3) \in (\mathbb{Z}\backslash\{0\})^4$. Then, for the local factor $\tau_p(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p))$, one has*

$$\tau_p(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)) = \det(1 - p^{-1}\,\mathrm{Frob}_p \mid \mathrm{Pic}(S_{\overline{\mathbb{Q}}})^{I_p}) \cdot \lim_{r\to\infty} \sum_{\nu_0,\ldots,\nu_3=0}^{r} \frac{z^{(r)}_{p^{3\nu_0}a_0,\,\ldots\,,p^{3\nu_3}a_3} \cdot \varphi(p^{r-\nu_0}) \cdot \ldots \cdot \varphi(p^{r-\nu_3})}{p^{3r} \cdot \varphi(p^r)^4}\,.$$

Proof. [PT, Corollary 3.5] implies that

$$\tau_p(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)) = \det(1 - p^{-1}\,\mathrm{Frob}_p \mid \mathrm{Pic}(S_{\overline{\mathbb{Q}}})^{I_p}) \cdot \lim_{r\to\infty} \sum_{\nu_0,\ldots,\nu_3=0}^{r} \frac{\#N^{(r)}_{\nu_0,\ldots,\nu_3;a_0,\ldots,a_3}}{p^{3r}}\,.$$

Lemma 2.3.8 yields the assertion. $\square$

2.3.10. Proposition. — *Let $(a_0, \ldots, a_3) \in (\mathbb{Z}\backslash\{0\})^4$. Then, for each ε such that $0 < \varepsilon < \frac{1}{3}$, one has*

$$\tau_p(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)) \leq \Big(1 + \frac{1}{p}\Big)^7 \cdot 3\Big(\frac{1}{1 - \frac{1}{p^{1-3\varepsilon}}}\Big)\Big(\frac{1}{1 - \frac{1}{p^{\varepsilon}}}\Big)^3 \cdot (a_0^{(p)}a_1^{(p)}a_2^{(p)})^{\frac{1-\varepsilon}{3}}(a_3^{(p)})^{\varepsilon}.$$

Proof. We use the formula from Corollary 2.3.9. The eigenvalues of the Frobenius on $\mathrm{Pic}(S_{\overline{\mathbb{Q}}})^{I_p}$ are all roots of unity. Therefore, the first factor is at most $(1 + 1/p)^7$. Further, by Lemma 2.3.6,

$$\begin{aligned} z^{(r)}_{p^{3\nu_0}a_0,\,\ldots\,,p^{3\nu_3}a_3}/p^{3r} &\leq 3\gcd\nolimits_p(p^{3\nu_0}a_0, \ldots, p^{3\nu_3}a_3) \\ &= 3\gcd(p^{3\nu_0}a_0^{(p)}, \ldots, p^{3\nu_3}a_3^{(p)}). \end{aligned}$$

Writing $k_i := \nu_p(a_i) = \nu_p(a_i^{(p)})$, we see

$$\begin{aligned} z^{(r)}_{p^{3\nu_0}a_0,\,\ldots\,,p^{3\nu_3}a_3}/p^{3r} &\leq 3\gcd(p^{3\nu_0+k_0}, \ldots, p^{3\nu_3+k_3}) \\ &= 3p^{\min\{3\nu_0+k_0,\,\ldots\,,3\nu_3+k_3\}}. \end{aligned}$$

We estimate the minimum by a weighted arithmetic mean with weights $\frac{1-\varepsilon}{3}$, $\frac{1-\varepsilon}{3}$, $\frac{1-\varepsilon}{3}$, and ε,

$$\begin{aligned}\min\{3\nu_0 + k_0, \ldots, 3\nu_3 + k_3\} &\leq \frac{1-\varepsilon}{3} \cdot (3\nu_0 + k_0) + \frac{1-\varepsilon}{3} \cdot (3\nu_1 + k_1) \\ &\qquad + \frac{1-\varepsilon}{3} \cdot (3\nu_2 + k_2) + \varepsilon(3\nu_3 + k_3) \\ &= (1-\varepsilon)(\nu_0 + \nu_1 + \nu_2) + 3\varepsilon\nu_3 \\ &\qquad + \frac{1-\varepsilon}{3}(k_0 + k_1 + k_2) + \varepsilon k_3 \,.\end{aligned}$$

This shows

$$\begin{aligned} z^{(r)}_{p^{3\nu_0}a_0,\ldots,p^{3\nu_3}a_3}/p^{3r} &\leq 3p^{(1-\varepsilon)(\nu_0+\nu_1+\nu_2)+3\varepsilon\nu_3+\frac{1-\varepsilon}{3}(k_0+k_1+k_2)+\varepsilon k_3} \\ &= 3p^{(1-\varepsilon)(\nu_0+\nu_1+\nu_2)+3\varepsilon\nu_3} \cdot (a_0^{(p)}a_1^{(p)}a_2^{(p)})^{\frac{1-\varepsilon}{3}}(a_3^{(p)})^{\varepsilon}.\end{aligned}$$

We may therefore write

$$\begin{aligned}\tau_p(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)) \leq \Big(1 + \frac{1}{p}\Big)^7 \cdot 3(a_0^{(p)}a_1^{(p)}a_2^{(p)})^{\frac{1-\varepsilon}{3}}(a_3^{(p)})^{\varepsilon} \\ \cdot \lim_{r\to\infty} \sum_{\nu_0,\ldots,\nu_3=0}^{r} \frac{p^{(1-\varepsilon)(\nu_0+\nu_1+\nu_2)+3\varepsilon\nu_3} \cdot \varphi(p^{r-\nu_0}) \cdot \ldots \cdot \varphi(p^{r-\nu_3})}{\varphi(p^r)^4} \,.\end{aligned}$$

Here, the term under the limit is precisely the product of three copies of the finite sum

$$\sum_{\nu=0}^{r} \frac{p^{(1-\varepsilon)\nu} \cdot \varphi(p^{r-\nu})}{\varphi(p^r)} = \sum_{\nu=0}^{r-1} \frac{1}{(p^{\varepsilon})^{\nu}} + \frac{p}{p-1}\frac{1}{(p^{\varepsilon})^{r}}$$

and one copy of the finite sum

$$\sum_{\nu=0}^{r} \frac{p^{3\varepsilon\nu} \cdot \varphi(p^{r-\nu})}{\varphi(p^r)} = \sum_{\nu=0}^{r-1} \frac{1}{(p^{1-3\varepsilon})^{\nu}} + \frac{p}{p-1}\frac{1}{(p^{1-3\varepsilon})^{r}} \,.$$

For $r \to \infty$, geometric series do appear while the additional summands tend to zero. □

2.3.11. Remark. — The constants

$$C_p^{(\varepsilon)} := \Big(1 + \frac{1}{p}\Big)^7 \cdot 3\Big(\frac{1}{1 - \frac{1}{p^{1-3\varepsilon}}}\Big)\Big(\frac{1}{1 - \frac{1}{p^{\varepsilon}}}\Big)^3$$

are clearly not optimal in any sense. Note, in particular, that we did not put much effort into the bound for $\det(1 - p^{-1}\,\mathrm{Frob}_p \mid \mathrm{Pic}(S_{\overline{\mathbb{Q}}})^{I_p})$.

However, and this is what is important for our application, we clearly have that $C_p^{(\varepsilon)}$ is bounded for $p \to \infty$, say $C_p^{(\varepsilon)} \leq C^{(\varepsilon)}$. We do not know of an improvement which would make the product $\prod_p C_p^{(\varepsilon)}$ converge.

2.3.12. Proposition. — *For each ε such that $0 < \varepsilon < \frac{1}{3}$, there exists a constant c such that*

$$\prod_{p \text{ prime}} \tau_p(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)) \leq c \cdot |a_0 \cdot \ldots \cdot a_3|^{\frac{1}{3}-\frac{\varepsilon}{8}} \cdot \prod_{p \text{ prime}} \min_{i=0,\ldots,3} \|a_i\|_p^{\frac{1}{3}-\varepsilon}$$

for all $(a_0, \ldots, a_3) \in (\mathbb{Z}\backslash\{0\})^4$.

Proof. The product over all primes of good reduction is bounded by virtue of Lemma 2.3.1 above. It, therefore, remains to show that

$$\prod_{\substack{p \text{ prime} \\ p|3a_0\cdots a_3}} \tau_p(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)) \leq c \cdot |a_0 \cdot \ldots \cdot a_3|^{\frac{1}{3}-\frac{\varepsilon}{8}} \cdot \prod_{p \text{ prime}} \min_{i=0,\ldots,3} \|a_i\|_p^{\frac{1}{3}-\varepsilon}.$$

For this, by Proposition 2.3.10, we have at first

$$\begin{aligned}\tau_p(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)) &\leq C_p^{(\varepsilon)} \cdot (a_0^{(p)} a_1^{(p)} a_2^{(p)})^{\frac{1}{3}-\frac{\varepsilon}{4}} \cdot (a_3^{(p)})^{\frac{3}{4}\varepsilon} \\ &= C_p^{(\varepsilon)} \cdot (a_0^{(p)} a_1^{(p)} a_2^{(p)} a_3^{(p)})^{\frac{1}{3}-\frac{\varepsilon}{4}} \cdot (a_3^{(p)})^{-\frac{1}{3}+\varepsilon}.\end{aligned}$$

Here, the indices $0, \ldots, 3$ are interchangeable. Hence, it is even allowed to write

$$\begin{aligned}\tau_p(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)) &\leq C_p^{(\varepsilon)} \cdot (a_0^{(p)} a_1^{(p)} a_2^{(p)} a_3^{(p)})^{\frac{1}{3}-\frac{\varepsilon}{4}} \cdot (\max_i a_i^{(p)})^{-\frac{1}{3}+\varepsilon} \\ &= C_p^{(\varepsilon)} \cdot (a_0^{(p)} a_1^{(p)} a_2^{(p)} a_3^{(p)})^{\frac{1}{3}-\frac{\varepsilon}{4}} \cdot \min_i \|a_i\|_p^{\frac{1}{3}-\varepsilon}.\end{aligned}$$

Now, we multiply over all prime divisors of $a_0 \cdot \ldots \cdot a_3$. Thereby, on the right hand side, we may twice write the product over all primes since the two rightmost factors are equal to one for $p \nmid 3a_0 \cdot \ldots \cdot a_3$, anyway.

$$\begin{aligned}\prod_{\substack{p \text{ prime} \\ p|3a_0\cdots a_3}} \tau_p(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)) &\leq \prod_{\substack{p \text{ prime} \\ p|3a_0\cdots a_3}} C_p^{(\varepsilon)} \cdot \prod_{p \text{ prime}} (a_0^{(p)} a_1^{(p)} a_2^{(p)} a_3^{(p)})^{\frac{1}{3}-\frac{\varepsilon}{4}} \cdot \prod_{p \text{ prime}} \min_{i=0,\ldots,3} \|a_i\|_p^{\frac{1}{3}-\varepsilon} \\ &= \prod_{\substack{p \text{ prime} \\ p|3a_0\cdots a_3}} C_p^{(\varepsilon)} \cdot |a_0 \cdot \ldots \cdot a_3|^{\frac{1}{3}-\frac{\varepsilon}{4}} \cdot \prod_{p \text{ prime}} \min_{i=0,\ldots,3} \|a_i\|_p^{\frac{1}{3}-\varepsilon}\end{aligned}$$

when we observe that $\prod_p a^{(p)} = |a|$. Further, we have $C_p^{(\varepsilon)} \leq C^{(\varepsilon)}$ and, by [Na, Theorem 7.2] together with [Na, Section 7.1, Exercise 7],

$$\prod_{\substack{p \text{ prime} \\ p|3a_0\cdots a_3}} C^{(\varepsilon)} \leq c \cdot |3a_0 \cdot \ldots \cdot a_3|^{\frac{\varepsilon}{8}}.$$

We finally estimate $3^{\frac{\varepsilon}{8}}$ by a constant. The assertion follows. □

2.4. An estimate for the factor at the infinite place

2.4.1. Proposition. — *For real numbers* $0 < b_0 \leq b_1 \leq b_2 \leq b_3$*, we have*

$$\int_{\substack{CS^{(1,\ldots,1)}(\mathbb{R}) \\ |x_0|\leq b_0,\ldots,|x_3|\leq b_3}} \omega_{\text{Leray}}^{CS^{(1,\ldots,1)}(\mathbb{R})} \leq \Big(64 + \frac{64}{3}\log 3 + \frac{1}{3}\sqrt[3]{3}\,\omega_2\Big) b_0 + 64 b_0 \log\frac{b_1}{b_0}$$

where ω_2 *is the two-dimensional hypersurface measure of the* l_3*-unit sphere*

$$S^2 := \{\, (x_1, x_2, x_3) \in \mathbb{R}^3 \mid |x_1|^3 + |x_2|^3 + |x_3|^3 = 1 \,\}.$$

Proof. *First step.* We cover the domain of integration by 25 sets as follows. We put

$$R_0 := [-b_0, b_0]^4 \cap CS^{(1,\dots,1)}(\mathbb{R})\,.$$

Further, for each $\sigma \in S_4$, we set

$$R_\sigma := \{\, (x_0, \dots, x_3) \in \mathbb{R}^4 \mid |x_{\sigma(0)}| \leq \dots \leq |x_{\sigma(3)}|,\ |x_i| \leq b_i,\ \text{and } b_0 \leq |x_{\sigma(3)}| \,\} \\ \cap CS^{(1,\dots,1)}(\mathbb{R})\,.$$

Second step. One has $\int_{R_\sigma} \omega_{\text{Leray}}^{CS^{(1,\dots,1)}(\mathbb{R})} \leq \int_{R_{\text{id}}} \omega_{\text{Leray}}^{CS^{(1,\dots,1)}(\mathbb{R})}$ for every $\sigma \in S_4$.

Consider the map $i_\sigma \colon \mathbb{R}^4 \to \mathbb{R}^4$ given by $(x_0, \dots, x_3) \mapsto (x_{\sigma(0)}, \dots, x_{\sigma(3)})$. Since $CS^{(1,\dots,1)}(\mathbb{R})$ is defined by a symmetric cubic form, it is invariant under i_σ. We claim that

$$i_\sigma(R_\sigma) \subseteq R_{\text{id}}\,.$$

Indeed, let $(x_0, \dots, x_3) \in R_\sigma$. Then, $i_\sigma(x_0, \dots, x_3) = (x_{\sigma(0)}, \dots, x_{\sigma(3)})$ has the properties $|x_{\sigma(0)}| \leq \ldots \leq |x_{\sigma(3)}|$ and $b_0 \leq |x_{\sigma(3)}|$. In order to show $i_\sigma(x_0, \dots, x_3) \in R_{\text{id}}$, all we need to verify is $|x_{\sigma(i)}| \leq b_i$ for $i = 0, \dots, 3$.

For this, we use that the b_i are sorted. We have $|x_{\sigma(3)}| \leq b_{\sigma(3)} \leq b_3$. Further, $|x_{\sigma(2)}| \leq b_{\sigma(2)}$ and $|x_{\sigma(2)}| \leq |x_{\sigma(3)}| \leq b_{\sigma(3)}$ one of which is at most equal to b_2. Similarly, $|x_{\sigma(1)}| \leq b_{\sigma(1)}$, $|x_{\sigma(1)}| \leq |x_{\sigma(2)}| \leq b_{\sigma(2)}$, and $|x_{\sigma(1)}| \leq |x_{\sigma(3)}| \leq b_{\sigma(3)}$, the smallest of which is not larger than b_1. Finally, $|x_{\sigma(0)}| \leq b_{\sigma(0)}$, $|x_{\sigma(0)}| \leq |x_{\sigma(1)}| \leq b_{\sigma(1)}$, $|x_{\sigma(0)}| \leq |x_{\sigma(2)}| \leq b_{\sigma(2)}$, and $|x_{\sigma(0)}| \leq |x_{\sigma(3)}| \leq b_{\sigma(3)}$. This shows $|x_{\sigma(0)}| \leq b_0$.

Since $x_0^3 + \ldots + x_3^3$ is a symmetric form, the Leray measure on $CS^{(1,\dots,1)}(\mathbb{R})$ is invariant under the canonical operation of S_4 on $CS^{(1,\dots,1)}(\mathbb{R}) \subset \mathbb{R}^4$. Therefore, we have $(i_\sigma)_* \omega_{\text{Leray}}^{CS^{(1,\dots,1)}(\mathbb{R})} = \omega_{\text{Leray}}^{CS^{(1,\dots,1)}(\mathbb{R})}$ for each $\sigma \in S_4$.

Altogether,

$$\int\limits_{R_\sigma} \omega_{\text{Leray}}^{CS^{(1,\dots,1)}(\mathbb{R})} \leq \int\limits_{i_\sigma^{-1}(R_{\text{id}})} \omega_{\text{Leray}}^{CS^{(1,\dots,1)}(\mathbb{R})} = \int\limits_{R_{\text{id}}} (i_\sigma)_*\, \omega_{\text{Leray}}^{CS^{(1,\dots,1)}(\mathbb{R})} = \int\limits_{R_{\text{id}}} \omega_{\text{Leray}}^{CS^{(1,\dots,1)}(\mathbb{R})}.$$

Third step. We have $\int_{R_0} \omega_{\text{Leray}}^{CS^{(1,\dots,1)}(\mathbb{R})} \leq \frac{1}{3}\sqrt[3]{3}\, \omega_2 b_0$.

By definition,

$$\begin{aligned}
\int\limits_{R_0} \omega_{\text{Leray}}^{CS^{(1,\dots,1)}(\mathbb{R})} &= \frac{1}{3} \int\limits_{R_0} \frac{1}{x_3^2}\, dx_0 \wedge dx_1 \wedge dx_2 \\
&= \frac{1}{3} \iiint\limits_{\pi(R_0)} \frac{1}{(x_0^3 + x_1^3 + x_2^3)^{2/3}}\, dx_0\, dx_1\, dx_2
\end{aligned}$$

where $\pi \colon CS^{(1,\dots,1)}(\mathbb{R}) \to \mathbb{R}^3$, $(x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, x_2)$, denotes the projection to the first three coordinates.

We enlarge the domain of integration to

$$R' := \{\, (x_1, x_2, x_3) \in \mathbb{R}^3 \mid |x_0|^3 + |x_1|^3 + |x_2|^3 \leq 3b_0^3 \,\}\,.$$

Then, by homogeneity, we see

$$\iiint_{R'} \frac{1}{(x_0^3 + x_1^3 + x_2^3)^{2/3}}\, dx_0\, dx_1\, dx_2 = \omega_2 \cdot \int_0^{\sqrt[3]{3}b_0} \frac{1}{r^2} \cdot r^2\, dr = \omega_2 \cdot \sqrt[3]{3} b_0\, .$$

Fourth step. We have $\int_{R_{\rm id}} \omega_{\rm Leray}^{CS^{(1,\ldots,1)}(\mathbb{R})} \leq (\frac{8}{3} + \frac{8}{9} \log 3) b_0 + \frac{8}{3} b_0 \log \frac{b_1}{b_0}$.

Observe $|x_3| = \left|\sqrt[3]{x_0^3 + x_1^3 + x_2^3}\right| \leq \sqrt[3]{|x_0|^3 + |x_1|^3 + |x_2|^3}$. For $(x_0, \ldots, x_3) \in R_{\rm id}$, this implies $|x_3| \leq \sqrt[3]{3}\, |x_2|$ and $|x_2| \geq b_0/\sqrt[3]{3}$. We find

$$\begin{aligned}
\int_{R_{\rm id}} \omega_{\rm Leray}^{CS^{(1,\ldots,1)}(\mathbb{R})} &= \frac{1}{3} \int_{R_{\rm id}} \frac{1}{x_3^2}\, dx_0 \wedge dx_1 \wedge dx_2 \\
&\leq \frac{1}{3} \int_{R_{\rm id}} \frac{1}{x_2^2}\, dx_0 \wedge dx_1 \wedge dx_2 \\
&< \frac{1}{3} \int_{-b_0}^{b_0} \int_{|x_1| \in [|x_0|, b_1]} \int_{\substack{|x_2| \geq b_0/\sqrt[3]{3} \\ |x_2| \geq |x_1|}} \frac{1}{x_2^2}\, dx_2\, dx_1\, dx_0 \\
&\leq \frac{1}{3} \int_{-b_0}^{b_0} \int_{|x_1| \in [|x_0|, b_1]} \frac{2}{\max\{b_0/\sqrt[3]{3}, |x_1|\}}\, dx_1\, dx_0 \\
&\leq \frac{2}{3} \left[\int_{-b_0}^{b_0} \int_{|x_1| \in [|x_0|, b_0/\sqrt[3]{3}]} \frac{\sqrt[3]{3}}{b_0}\, dx_1\, dx_0 + \int_{-b_0}^{b_0} \int_{|x_1| \in [b_0/\sqrt[3]{3}, b_1]} \frac{1}{|x_1|}\, dx_1\, dx_0 \right] \\
&\leq \frac{2}{3} \cdot \frac{4b_0^2}{\sqrt[3]{3}} \cdot \frac{\sqrt[3]{3}}{b_0} + \frac{2}{3} \int_{-b_0}^{b_0} 2 \log \frac{\sqrt[3]{3} b_1}{b_0}\, dx_0 \\
&= \frac{8}{3} b_0 + \frac{8}{3} b_0 \log \frac{\sqrt[3]{3} b_1}{b_0} \\
&= \Big(\frac{8}{3} + \frac{8}{9} \log 3\Big) b_0 + \frac{8}{3} b_0 \log \frac{b_1}{b_0}.
\end{aligned}$$

□

2.4.2. Corollary. —— *For every $\varepsilon > 0$, there exists a constant c such that*

$$\tau_\infty(S^{(a_0,\ldots,a_3)}(\mathbb{R})) \leq c \cdot |a_0 \cdot \ldots \cdot a_3|^{-\frac{1}{3}+\varepsilon} \cdot \min_{i=0,\ldots,3} \|a_i\|_\infty^{\frac{1}{3}}$$

for each $(a_0, \ldots, a_3) \in (\mathbb{Z} \backslash \{0\})^4$.

Proof. Our first claim is

$$\tau_\infty(S^{(a_0,\ldots,a_3)}(\mathbb{R})) = \frac{1}{2\sqrt[3]{|a_0 \cdot \ldots \cdot a_3|}} \int_{\substack{CS^{(1,\ldots,1)}(\mathbb{R}) \\ |x_0| \leq \sqrt[3]{|a_0|}, \ldots, |x_3| \leq \sqrt[3]{|a_3|}}} \omega_{\rm Leray}^{CS^{(1,\ldots,1)}(\mathbb{R})}.$$

Indeed, according to the definition of $\tau_\infty(S^{(a_0,\ldots,a_3)}(\mathbb{R}))$, we need to show

$$\frac{1}{6\,|a_0|}\int\limits_{\substack{CS^{(a_0,\ldots,a_3)}(\mathbb{R})\\ |x_0|\leq 1,\ldots,|x_3|\leq 1}}\frac{1}{x_0^2}\,dx_1\wedge dx_2\wedge dx_3=\frac{1}{6\sqrt[3]{|a_0\cdot\ldots\cdot a_3|}}\int\limits_{\substack{CS^{(1,\ldots,1)}(\mathbb{R})\\ |X_0|\leq\sqrt[3]{|a_0|},\ldots,|X_3|\leq\sqrt[3]{|a_3|}}}\frac{1}{X_0^2}\,dX_1\wedge dX_2\wedge dX_3\,.$$

For that, consider the linear mapping $l\colon CS^{(a_0,\ldots,a_3)}(\mathbb{R})\to CS^{(1,\ldots,1)}(\mathbb{R})$ given by

$$(x_0,\ldots,x_3)\mapsto(\sqrt[3]{a_0}x_0,\ldots,\sqrt[3]{a_3}x_3)\,.$$

Then,

$$l^*\Big(\frac{1}{X_0^2}\,dX_1\wedge dX_2\wedge dX_3\Big)=\frac{\sqrt[3]{a_1a_2a_3}}{a_0^{2/3}}\frac{1}{x_0^2}\,dx_1\wedge dx_2\wedge dx_3.$$

When we take into consideration that orientations are chosen in such a way that both integrals are positive, this immediately yields the claim.

To obtain the asserted inequality, we assume without restriction that $|a_0|\leq\ldots\leq|a_3|$. Then, Proposition 2.4.1 shows that, for certain explicit positive constants c_1 and c_2,

$$\begin{aligned}\tau_\infty(S^{(a_0,\ldots,a_3)}(\mathbb{R}))&\leq|a_0\cdot\ldots\cdot a_3|^{-\frac{1}{3}}\cdot\left(c_1|a_0|^{\frac{1}{3}}+c_2|a_0|^{\frac{1}{3}}\log\sqrt[3]{\frac{|a_1|}{|a_0|}}\right)\\&=|a_0\cdot\ldots\cdot a_3|^{-\frac{1}{3}}\cdot|a_0|^{\frac{1}{3}}\Big(c_1+\frac{1}{3}\,c_2\log\frac{|a_1|}{|a_0|}\Big)\\&\leq|a_0\cdot\ldots\cdot a_3|^{-\frac{1}{3}}\cdot\min_{i=0,\ldots,3}\|a_i\|_\infty^{\frac{1}{3}}\cdot\Big(c_1+\frac{1}{3}\,c_2\log|a_0\cdot\ldots\cdot a_3|\Big).\end{aligned}$$

There is a constant c such that $c_1+\frac{1}{3}c_2\log|a_0\cdot\ldots\cdot a_3|\leq c|a_0\cdot\ldots\cdot a_3|^\varepsilon$ for every $(a_0,\ldots,a_3)\in(\mathbb{Z}\backslash\{0\})^4$. □

2.5. The Tamagawa number

2.5.1. Proposition. — *For every $\varepsilon>0$, there exists a constant $C>0$ such that*

$$\frac{1}{\tau^{(a_0,\ldots,a_3)}}\geq C\cdot\frac{\mathrm{H}_{\text{naive}}\,(\frac{1}{a_0}:\ldots:\frac{1}{a_3})^{\frac{1}{3}}}{|a_0\cdot\ldots\cdot a_3|^\varepsilon}$$

for each $(a_0,\ldots,a_3)\in(\mathbb{Z}\backslash\{0\})^4$.

Proof. We may assume that ε is small, say $\varepsilon<\frac{2}{3}$. Then, immediately from the definition of $\tau^{(a_0,\ldots,a_3)}$, we have

$$\begin{aligned}&\tau^{(a_0,\ldots,a_3)}\\&=\alpha(S^{(a_0,\ldots,a_3)})\cdot\beta(S^{(a_0,\ldots,a_3)})\cdot\lim_{s\to1}(s-1)^tL\,(s,\chi_{\mathrm{Pic}(S^{(a_0,\ldots,a_3)}_{\overline{\mathbb{Q}}})})\cdot\tau_H\,(S^{(a_0,\ldots,a_3)}(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}})\\&\leq\alpha(S^{(a_0,\ldots,a_3)})\cdot\beta(S^{(a_0,\ldots,a_3)})\cdot\lim_{s\to1}(s-1)^tL\,(s,\chi_{\mathrm{Pic}(S^{(a_0,\ldots,a_3)}_{\overline{\mathbb{Q}}})})\cdot\tau_H\,(S^{(a_0,\ldots,a_3)}(\mathbb{A}_\mathbb{Q}))\\&=\alpha(S^{(a_0,\ldots,a_3)})\cdot\beta(S^{(a_0,\ldots,a_3)})\cdot\lim_{s\to1}(s-1)^tL\,(s,\chi_{\mathrm{Pic}(S^{(a_0,\ldots,a_3)}_{\overline{\mathbb{Q}}})})\cdot\prod_{\nu\in\mathrm{Val}(\mathbb{Q})}\tau_\nu(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_\nu)).\end{aligned}$$

Let us collect estimates for the factors. First, by Proposition 2.2.4, we have

$$\lim_{s\to 1}(s-1)^t L(s, \chi_{\mathrm{Pic}(S^{(a_0,\ldots,a_3)}_{\overline{\mathbb{Q}}})}) < c_1 \cdot |a_0 \cdot \ldots \cdot a_3|^{\frac{\varepsilon}{16}}$$

for a certain constant c_1. Further, Proposition 2.3.12 yields

$$\prod_{p \text{ prime}} \tau_p(S^{(a_0,\ldots,a_3)}(\mathbb{Q}_p)) \leq c_2 \cdot |a_0 \cdot \ldots \cdot a_3|^{\frac{1}{3}-\frac{\varepsilon}{16}} \cdot \prod_{p \text{ prime}} \min_{i=0,\ldots,3} \|a_i\|_p^{\frac{1}{3}-\frac{\varepsilon}{2}}.$$

Finally, Corollary 2.4.2 shows

$$\tau_\infty(S^{(a_0,\ldots,a_3)}(\mathbb{R})) \leq c \cdot |a_0 \cdot \ldots \cdot a_3|^{-\frac{1}{3}+\frac{\varepsilon}{2}} \cdot \min_{i=0,\ldots,3} \|a_i\|_\infty^{\frac{1}{3}}.$$

We assert that the three inequalities together imply the following estimate for Peyre's constant $\tau^{(a_0,\ldots,a_3)} = \tau(S^{(a_0,\ldots,a_3)})$,

$$\tau^{(a_0,\ldots,a_3)} \leq c_3 \cdot |a_0 \cdot \ldots \cdot a_3|^{\frac{\varepsilon}{2}} \cdot \prod_{p \text{ prime}} \min_{i=0,\ldots,3} \|a_i\|_p^{\frac{1}{3}} \cdot \min_{i=0,\ldots,3} \|a_i\|_\infty^{\frac{1}{3}} \cdot \prod_{p \text{ prime}} \Big[\min_{i=0,\ldots,3} \|a_i\|_p \Big]^{-\frac{\varepsilon}{2}}.$$

Indeed, this is trivial in the case $\tau^{(a_0,\ldots,a_3)} = 0$. Otherwise, $S^{(a_0,\ldots,a_3)}$ has an adelic point. Lemmata 2.1.5 and 2.1.3 show that the factors α and β are bounded from above by constants. By consequence,

$$\begin{aligned}
\frac{1}{\tau^{(a_0,\ldots,a_3)}} &\geq \frac{1}{c_3} \cdot \frac{\prod\limits_{p \text{ prime}} \Big[\min\limits_{i=0,\ldots,3} \|a_i\|_p \Big]^{-\frac{1}{3}} \cdot \Big[\min\limits_{i=0,\ldots,3} \|a_i\|_\infty \Big]^{-\frac{1}{3}}}{|a_0 \cdot \ldots \cdot a_3|^{\frac{\varepsilon}{2}} \cdot \prod\limits_{p \text{ prime}} \big[\min\limits_{i=0,\ldots,3} \|a_i\|_p \big]^{-\frac{\varepsilon}{2}}} \\
&= \frac{1}{c_3} \cdot \frac{\prod\limits_{p \text{ prime}} \max\limits_{i=0,\ldots,3} \Big\| \frac{1}{a_i} \Big\|_p^{\frac{1}{3}} \cdot \max\limits_{i=0,\ldots,3} \Big\| \frac{1}{a_i} \Big\|_\infty^{\frac{1}{3}}}{|a_0 \cdot \ldots \cdot a_3|^{\frac{\varepsilon}{2}} \cdot \prod\limits_{p \text{ prime}} \Big[\max\limits_{i=0,\ldots,3} a_i^{(p)} \Big]^{\frac{\varepsilon}{2}}} \\
&= \frac{1}{c_3} \cdot \frac{\mathrm{H}_{\text{naive}}(\frac{1}{a_0} : \ldots : \frac{1}{a_3})^{\frac{1}{3}}}{|a_0 \cdot \ldots \cdot a_3|^{\frac{\varepsilon}{2}} \cdot \prod\limits_{p \text{ prime}} \Big[\max\limits_{i=0,\ldots,3} a_i^{(p)} \Big]^{\frac{\varepsilon}{2}}} .
\end{aligned}$$

It is obvious that $\max\limits_{i=0,\ldots,3} a_i^{(p)} \leq |a_0^{(p)} \cdot \ldots \cdot a_3^{(p)}|$ and $\prod\limits_{p \text{ prime}} |a_0^{(p)} \cdot \ldots \cdot a_3^{(p)}| = |a_0 \cdot \ldots \cdot a_3|$. This shows

$$\begin{aligned}
\frac{1}{\tau^{(a_0,\ldots,a_3)}} &\geq \frac{1}{c_3} \cdot \frac{\mathrm{H}_{\text{naive}}(\frac{1}{a_0} : \ldots : \frac{1}{a_3})^{\frac{1}{3}}}{|a_0 \cdot \ldots \cdot a_3|^{\frac{\varepsilon}{2}} \cdot |a_0 \cdot \ldots \cdot a_3|^{\frac{\varepsilon}{2}}} \\
&= \frac{1}{c_3} \cdot \frac{\mathrm{H}_{\text{naive}}(\frac{1}{a_0} : \ldots : \frac{1}{a_3})^{\frac{1}{3}}}{|a_0 \cdot \ldots \cdot a_3|^{\varepsilon}} .
\end{aligned}$$

□

2.5.2. Lemma. —— *Let* $(a_0 : \ldots : a_3) \in \mathbf{P}^3(\mathbb{Q})$ *be any point such that* $a_0 \neq 0, \ldots, a_3 \neq 0$. *Then,*

$$\mathrm{H}_{\text{naive}}(a_0 : \ldots : a_3) \leq \mathrm{H}_{\text{naive}}(\tfrac{1}{a_0} : \ldots : \tfrac{1}{a_3})^3 .$$

Proof. First, observe that $(a_0 : \ldots : a_3) \mapsto (\frac{1}{a_0} : \ldots : \frac{1}{a_3})$ is a well-defined map. Hence, we may assume without restriction that $a_0, \ldots, a_3 \in \mathbb{Z}$ and $\gcd(a_0, \ldots, a_3) = 1$. This yields $H_{\text{naive}}(a_0 : \ldots : a_3) = \max_{i=0,\ldots,3} |a_i|$.

On the other hand, $(\frac{1}{a_0} : \ldots : \frac{1}{a_3}) = (a_1a_2a_3 : \ldots : a_0a_1a_2)$. Consequently,

$$H_{\text{naive}}\,(\tfrac{1}{a_0} : \ldots : \tfrac{1}{a_3}) \leq [\max_{i=0,\ldots,3} |a_i|]^3 = H_{\text{naive}}(a_0 : \ldots : a_3)^3 .$$

From this, the asserted inequality emerges when the roles of a_i and $\frac{1}{a_i}$ are interchanged. $\square$

2.5.3. Corollary. —— *Let* $a_0, \ldots, a_3 \in \mathbb{Z}$ *such that* $\gcd(a_0, \ldots, a_3) = 1$. *Then,*

$$|a_0 \cdot \ldots \cdot a_3| \leq H_{\text{naive}}\,(\tfrac{1}{a_0} : \ldots : \tfrac{1}{a_3})^{12}.$$

Proof. Observe that $|a_0 \cdot \ldots \cdot a_3| \leq \max\limits_{i=0,\ldots,3} |a_i|^4 = H_{\text{naive}}(a_0 : \ldots : a_3)^4$ and apply Lemma 2.5.2. $\square$

2.5.4. Theorem. —— *For each* $\varepsilon > 0$, *there exists a constant* $C(\varepsilon) > 0$ *such that, for all* $(a_0, \ldots, a_3) \in (\mathbb{Z} \setminus \{0\})^4$,

$$\frac{1}{\tau^{(a_0,\ldots,a_3)}} \geq C(\varepsilon) \cdot H_{\text{naive}}(\tfrac{1}{a_0} : \ldots : \tfrac{1}{a_3})^{\frac{1}{3}-\varepsilon}.$$

Proof. We may assume that $\gcd(a_0, \ldots, a_3) = 1$. Then, by Proposition 2.5.1,

$$\frac{1}{\tau^{(a_0,\ldots,a_3)}} \geq C(\varepsilon) \cdot \frac{H_{\text{naive}}(\frac{1}{a_0} : \ldots : \frac{1}{a_3})^{\frac{1}{3}}}{|a_0 \cdot \ldots \cdot a_3|^{\frac{\varepsilon}{12}}} .$$

Corollary 2.5.3 yields $|a_0 \cdot \ldots \cdot a_3|^{\frac{\varepsilon}{12}} \leq H_{\text{naive}}(\frac{1}{a_0} : \ldots : \frac{1}{a_3})^{\varepsilon}$. $\square$

2.5.5. Corollary (Fundamental finiteness**).** —— *For each* $T > 0$, *there are only finitely many diagonal cubic surfaces* $S^{(a_0,\ldots,a_3)} \colon a_0x_0^3 + \ldots + a_3x_3^3 = 0$ *in* $\mathbf{P}^3_{\mathbb{Q}}$ *such that* $\tau^{(a_0,\ldots,a_3)} > T$.

Proof. This is an immediate consequence of the comparison to the naive height established in Theorem 2.5.4. $\square$

3. The varieties of Batyrev-Tschinkel

3.1. Lemma. —— *Let* m, n *be positive integers such that* $m \leq n+1$ *and* $\iota \colon \mathbf{P}^m \dashrightarrow \mathbf{P}^3$ *a surjective linear map. Then, there exists a constant* C *such that, for every* $(a_0 : \ldots : a_3) \in \mathbf{P}^3(\mathbb{Q})$,

$$\sum_{\substack{x \in \mathbf{P}^m(\mathbb{Q}) \\ \iota(x) = (a_0 : \ldots : a_3)}} \frac{1}{H^n_{\text{naive}}(x)} \leq C \cdot \frac{1}{H^{n-m+3}_{\text{naive}}(a_0 : \ldots : a_3)} .$$

Proof. An automorphism of $\mathbf{P}^m$ changes the naive height by a factor which is bounded. We may therefore suppose that ι is given by $(x_0 : \ldots : x_m) \mapsto (x_0 : \ldots : x_3)$. Further, assume $a_0, \ldots, a_3 \in \mathbb{Z}$ such that $\gcd(a_0, \ldots, a_3) = 1$. Finally, we write $H := H_{\text{naive}}(a_0 : \ldots : a_3)$.

Let $N \geq \mathrm{H}$ be an arbitrary integer. There are two ways a point $x = (x_0 : \ldots : x_m) \in \mathbf{P}^m(\mathbb{Q})$ such that $\iota(x) = (a_0 : \ldots : a_3)$ may have height exactly equal to N. Either, one of the coordinates $x_4, \ldots, x_m$ is equal to N. There are at most

$$2[\tfrac{N}{\mathrm{H}}](m-3)(2N+1)^{m-4} \leq \tfrac{C_1}{\mathrm{H}} N^{m-3}$$

such points. Or, one of the coordinates $x_0, \ldots, x_3$ is equal to N. This is possible only when $N = k\mathrm{H}$ is an exact multiple. Then, there are at most

$$(2N+1)^{m-3} \leq C_2 N^{m-3}$$

such points. All in all, we find the estimate

$$\sum_{\substack{x \in \mathbf{P}^m(\mathbb{Q}) \\ \iota(x) = (a_0 : \ldots : a_3)}} \frac{1}{\mathrm{H}^n_{\mathrm{naive}}(x)} \leq \frac{C_1}{\mathrm{H}} \sum_{N=\mathrm{H}}^{\infty} \frac{1}{N^{n-m+3}} + \frac{C_2}{\mathrm{H}^{n-m+3}} \sum_{k=1}^{\infty} \frac{1}{k^{n-m+3}} \leq \frac{C}{\mathrm{H}^{n-m+3}} \,.$$

The assumption $m \leq n+1$ assures that all the series occurring are convergent. $\square$

3.2. Proposition. — *Let* $\iota\colon \mathbf{P}^n \dashrightarrow (\mathbf{P}^3)^\vee$ *be a linear map. Suppose that either* $\dim \operatorname{im} \iota \geq 2$ *or* $n = 1$. *Then, the series*

$$\sum_{\substack{x \in \mathbf{P}^n(\mathbb{Q}) \\ S^{\iota(x)} \text{ non-singular} \\ \operatorname{rk} \operatorname{Pic}(S^{\iota(x)}) = 4}} \frac{1}{\mathrm{H}^n_{\mathrm{naive}}(x)}$$

is convergent.

Proof. Note that Picard rank 4 is the maximal value which is possible for a non-singular diagonal cubic surface. It occurs for $S^{(a_0 : \ldots : a_3)}$ if and only if all the quotients a_i/a_0 are perfect cubes in $\mathbb{Q}$. We will distinguish three cases.

First case. $\dim \operatorname{im} \iota = 3$.

There are at most $4(2N+1)^3$ quadruples $(a_0 : \ldots : a_3)$ of naive height N^3 such that all the quotients a_i/a_0 are perfect cubes. According to Lemma 3.1, the series to be considered is dominated by $\sum\limits_N 4(2N+1)^3 \frac{C}{(N^3)^3} \leq 108C \sum\limits_N \frac{1}{N^6}$ which converges.

Second case. $\dim \operatorname{im} \iota = 2$.

Then, ι is the restriction of a surjective linear map $\mathbf{P}^{n+1} \to \mathbf{P}^3$ to a hyperplane. Estimating very roughly, we find the convergent series $\sum\limits_N 4(2N+1)^3 \frac{C}{(N^3)^2} \leq 108C \sum\limits_N \frac{1}{N^3}$.

Third case. $\dim \operatorname{im} \iota = 1$.

Here, by assumption, $n = 1$. An automorphism of $\mathbf{P}^1$ changes the naive height by a factor which is bounded. Thus, without restriction, we may suppose that ι is given by

$$(x_0 : x_1) \mapsto (x_0 : x_1 : l_1(x_0, x_1) : l_2(x_0, x_1))$$

for two linear forms l_1, l_2. As $\mathrm{H}_{\mathrm{naive}}(x_0 : x_1 : l_1(x_0, x_1) : l_2(x_0, x_1)) \geq \mathrm{H}_{\mathrm{naive}}(x_0 : x_1)$, the contribution of $(x_0 : x_1) \in \mathbf{P}^1(\mathbb{Q})$ is estimated by $\frac{1}{\mathrm{H}_{\mathrm{naive}}(x_0 : x_1)}$. Further, we only consider pairs such that x_1/x_0 a perfect cube. There are $\leq 2(2N+1)$ such pairs $(x_0 : x_1)$ of naive height N^3. The series $\sum\limits_N 2(2N+1)\frac{1}{N^3} \leq 6 \sum\limits_N \frac{1}{N^2}$ converges. $\square$

3.3. Corollary (The Batyrev-Tschinkel varieties**).** — *Let $X \subset \mathbf{P}^n \times \mathbf{P}^3$ be a smooth hypersurface given by a bihomogeneous form of the shape*

$$\iota_0(x_0, \ldots, x_n)\, y_0^3 + \cdots + \iota_3(x_0, \ldots, x_n)\, y_3^3 \,.$$

Suppose that $\iota_0, \ldots, \iota_3$ are linear forms, not all proportional to each other. Then, the series

$$\sum_{\substack{x \in \mathbf{P}^n(\mathbb{Q}) \\ S^{\iota(x)} \text{ non-singular} \\ \operatorname{rk}\operatorname{Pic}(S^{\iota(x)})=4}} \frac{1}{\mathrm{H}^n_{\text{naive}}(x)}\, \tau(S^{\iota(x)})$$

converges. Here, $\iota\colon \mathbf{P}^n \dashrightarrow (\mathbf{P}^3)^\vee$ is the linear map defined by $\iota_0, \ldots, \iota_3$.

Proof. Theorem 2.5.4 immediately implies that the factors $\tau(S^{\iota(x)})$ are bounded. Further, as X is smooth [BT, Proposition 1.1], we have $\dim \operatorname{im} \iota = \min(n, 3)$. Thus, the assertion is a direct consequence of Lemma 3.2. □

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