ON PLANE QUARTICS WITH A GALOIS INVARIANT CAYLEY OCTAD

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Abstract. We describe a construction of plane quartics with prescribed Galois operation on the 28 bitangents, in the particular case of a Galois invariant Cayley octad. As an application, we solve the inverse Galois problem for degree two del Pezzo surfaces in the corresponding particular case.

1. Introduction

It is well known [Pl] that a nonsingular plane quartic curve C over an algebraically closed field of characteristic $\neq 2$ has exactly 28 bitangents. The same is still true if the base field is only separably closed, as is easily deduced from [Va, Theorem 1.6]. If C is defined over a separably non-closed field k then the bitangents are defined over a finite extension field l of k, which is normal and separable, and permuted by the Galois group $\mathrm{Gal}(l/k)$.

By far not every permutation in S_{28} may occur. It is well-known (cf., e.g., [PSV]) that the subgroup $G \subset S_{28}$ of all admissible permutations is independent of the choice of C and isomorphic to $\mathrm{Sp}_6(\mathbb{F}_2)$. A natural question arising is thus the following.

Question. Given a field k and a subgroup $g \subseteq G$, does there exist a nonsingular plane quartic C over k, for which the group homomorphism $\mathrm{Gal}(k^{\mathrm{sep}}/k) \to G \subset S_{28}$, given by the Galois operation on the 28 bitangents, has image g? This question depends only on the conjugacy class of the subgroup $g \subseteq G$.

Remark 1.1. At least when k is a global field, the Galois operation on the 28 bitangents and, in particular, the subgroup of G being its image, form important invariants concerning the arithmetic of the curve C. For interesting applications in the closely related situations of Del Pezzo surfaces of low degree, we advise the reader to consult Yu. I. Manin's book [Ma] or the article [Shi] of T. Shioda.

The group $G \cong \mathrm{Sp}_6(\mathbb{F}_2)$ is the simple group of order 1 451 520. It has 1369 conjugacy classes of subgroups. Among these, there are eight maximal subgroups, which are of indices 28, 36, 63, 120, 135, 315, 336, and 960, respectively.

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An example of a nonsingular plane quartic over $\mathbb{Q}$ such that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ operates on the bitangents via the full $\mathrm{Sp}_6(\mathbb{F}_2)$ has been constructed by T. Shioda [Shi, Section 3] in 1993 and, almost at the same time, by R. Erné [Er, Corollary 3]. Moreover, there is an obvious approach to construct examples for the groups contained in the index 28 subgroup. Indeed, in this case, there is a rational bitangent. One may start with a cubic surface with the right Galois operation [EJ15], blow-up a rational point, and use the connection between degree two del Pezzo surfaces and plane quartics [Ko96, Theorem 3.3.5], cf. the application discussed below (and mainly in Section 3). For the group U_{63} of index 63 and its subgroups, we gave a complete answer in our previous work [EJ17].

In this article, we deal with the subgroup $U_{36} \subset G$ of index 36 and the groups contained within. More precisely, we show the following result, which answers a more refined question than the one asked above.

Theorem 1.2. *Let an infinite field k of characteristic not 2, a normal and separable extension field l, and an injective group homomorphism*

$$i\colon \mathrm{Gal}(l/k) \hookrightarrow U_{36}$$

be given. Then there exists a nonsingular quartic curve C over k such that l is the field of definition of the 28 bitangents and each $\sigma \in \mathrm{Gal}(l/k)$ permutes the bitangents as described by $i(\sigma) \in G \subset S_{28}$.

Among the 1369 conjugacy classes of subgroups of $G \cong \mathrm{Sp}_6(\mathbb{F}_2)$, 296 are contained in U_{36}. However, 262 of these are also contained in U_{63}, such that they are covered by [EJ17], already, while 34 are new.

The subgroup U_{36} has a geometric meaning. The assumption that the image of the natural homomorphism $i\colon \mathrm{Gal}(k^{\mathrm{sep}}/k) \to G$ is contained in U_{36} expresses the fact that there exists a Galois invariant Cayley octad (cf. [Sen, §5], [PSV, §3], or [Do, Definition 6.3.4]) defining the quartic C. Moreover, one has an isomorphism of groups $U_{36} \cong S_8$ and U_{36} permutes the eight points forming the octad accordingly. We use Cayley octads for our proof of existence, which is completely constructive.

Remarks 1.3. i) Theorem 1.2 is clearly not true, in general, when k is a finite field. For example, there cannot be a nonsingular quartic curve over $\mathbb{F}_3$, all whose bitangents are $\mathbb{F}_3$-rational, simply because the projective plane contains only 13 $\mathbb{F}_3$-rational lines.

ii) We ignore about characteristic 2 in this article, as this case happens to be very different. Even over an algebraically closed field, a plane quartic cannot have more than seven bitangents [SV, p. 60].

As an application, one may answer the analogous question for degree two del Pezzo surfaces. The double cover of $\mathbf{P}^2$, ramified at a nonsingular quartic curve C is a del Pezzo surface of degree two. Here, considerations can be made that are very similar to the ones above. First of all, it is well known [Ma, Theorem 26.2.(iii)] that a del Pezzo surface S of degree two over an algebraically closed field contains exactly 56 exceptional curves, i.e. such of self-intersection number (-1). Again, the same is true when the base field is only separably closed. If S is defined over a separably non-closed field k then the exceptional curves are defined over a normal and separable finite extension field l of k and permuted by $\mathrm{Gal}(l/k)$. Once again, not every permutation in S_{56} may occur. The maximal subgroup $\widetilde{G} \subset S_{56}$ that respects the intersection pairing is isomorphic to the Weyl group $W(E_7)$ [Ma, Theorem 23.9].

Every bitangent of C is covered by exactly two of the exceptional curves of S. Thus, for the operation of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ on the 56 exceptional curves on S, there seem to be two independent conditions. On one hand, $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ must operate via a subgroup of $W(E_7) \cong \widetilde{G} \subset S_{56}$. On the other hand, the induced operation on the blocks of size two must take place via a subgroup of $\mathrm{Sp}_6(\mathbb{F}_2) \cong G \subset S_{28}$. In turns out, however, that there is an isomorphism $W(E_7)/Z \xrightarrow{\cong} \mathrm{Sp}_6(\mathbb{F}_2)$, for $Z \subset W(E_7)$ the centre, that makes the two conditions equivalent. Cf. [EJ17, Corollary 2.17].

The group $\widetilde{G} \cong W(E_7)$ already has 8074 conjugacy classes of subgroups. Two subgroups with the same image under the quotient map $p\colon \widetilde{G} \twoheadrightarrow \widetilde{G}/Z \xrightarrow{\cong} G$ correspond to del Pezzo surfaces of degree two that are quadratic twists of each other. Theorem 1.2 therefore extends word-by-word to del Pezzo surfaces of degree two and homomorphisms $\mathrm{Gal}(l/k) \hookrightarrow \widetilde{G}$ with image contained in $p^{-1}(U_{36})$.

Remark 1.4 (Conjugacy classes in G versus conjugacy classes in U_{36})*.* As noticed above, 296 of the conjugacy classes of subgroups of $G \cong \mathrm{Sp}_6(\mathbb{F}_2)$ are contained in $S_8 \cong U_{36} \subset G$. On the other hand, one may readily check that S_8 itself has precisely 296 conjugacy classes of subgroups. In other words, two subgroups $U_1, U_2 \subseteq U_{36}$ that are conjugate in G must be conjugate in U_{36}, already.

This result came to us originally as an experimental finding, and as quite a surprise. As it is pure group theory and thus somewhat off the main topic of this article, we provide a proof in an appendix.

Convention 1.5*.* In this article, by a *field,* we mean a field of characteristic $\neq 2$. For the convenience of the reader, the assumption on the characteristic will be repeated in the formulations of our final results, but not during the intermediate steps.

Computations. All computations are done using `magma` [BCP].

2. Cayley octads with prescribed Galois operation

All space quadrics over a field k form a $\mathbf{P}^9$, the generic quadric being

$$(T_0\,T_1\,T_2\,T_3)\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{pmatrix}\begin{pmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \end{pmatrix} = 0\,.$$

Among these, the singular quadrics are parametrised by the quartic $Q_s \subset \mathbf{P}^9$, given by $D_s = 0$, for

$$D_s := \det \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

Lemma 2.1. *The quartic Q_s is singular, the singular points corresponding exactly to the quadrics of rank $\leqslant 2$.*

Proof. This may be tested after base extension to the algebraic closure. Thus, let us suppose that $k = \overline{k}$. The quartic Q_s parametrises all singular quadrics, i.e. those of rank $\leqslant 3$. If the rank of a quadric q is exactly three then, after a linear transformation of coordinates, we may assume that $q = T_0^2 + T_1^2 + T_2^2$. At the point where $a_{00} = a_{11} = a_{22} = 1$ and all other coordinates vanish, the gradient of D_s is a_{33}. Thus, q corresponds to a regular point.

On the other hand, if the rank of q is $\leqslant 2$ then one may assume without restriction that $q = T_0^2 + cT_1^2$, for $c = 1$ or 0. At the point where $a_{00} = 1$, $a_{11} = c$, and all other coordinates vanish, the gradient of D_s is zero. The point is singular. $\square$

Let now q_1, q_2, and q_3 be three linearly independent quadratic forms, defined by the symmetric matrices M_1, M_2, and M_3. These span a net

$$\Lambda := (Z(u_1q_1 + u_2q_2 + u_3q_3))_{(u_1:u_2:u_3)\in\mathbf{P}^2}$$

of quadrics, which contains its singular members at the locus

$$C_\Lambda := Z(\det(u_1M_1 + u_2M_2 + u_3M_3)) \subset \mathbf{P}^2 .$$

This is a plane quartic.

Lemma 2.2. *Assume that the base locus of the net Λ consists of eight distinct points, no four of which are coplanar. Then the quartic C_Λ is nonsingular.*

Proof. Again, this may be tested after base change to the algebraic closure. The quartic C_Λ is then the intersection of Q_s with a two-dimensional linear subspace $E \subset \mathbf{P}^9$. Thus, there are two ways, in which C_Λ may become singular. Either, the linear subspace intersects the singular locus of Q_s or it meets Q_s somewhere tangentially.

By Lemma 2.1, the first option would mean that $\Lambda = \langle q_1, q_2, q_3 \rangle$ contains a quadric of rank $\leqslant 2$. Such a quadric, however, splits into two linear forms, so the base locus of Λ, given in $\mathbf{P}^3$ by $q_1 = q_2 = q_3 = 0$, is necessarily contained in the union of two planes. This contradicts our assumptions.

Thus, let us assume that E meets Q_s somewhere tangentially. Without restriction, the point of tangency corresponds to the space quadric $q = T_0^2+T_1^2+T_2^2$. The tangent hyperplane at the corresponding point on $Q_s \subset \mathbf{P}^9$ is given by $a_{33} = 0$, so that our assumption means that $\Lambda = \langle q, q_1, q_2 \rangle$ and both q_1 and q_2 have coefficient zero at T_3^2. But then the base locus of Λ contains $(0\!:\!0\!:\!0\!:\!1)$ at least as a double point, which is a contradiction, too. $\square$

Definition 2.3. A set $X \subset \mathbf{P}^3$ of eight points is called a *Cayley octad* if it is the base locus of a unique net of quadrics. In this case, we write Λ_X for the net of quadrics through X.

Remarks 2.4. i) Observe that eight points in general position in $\mathbf{P}^3$ define only a pencil of quadrics. Thus, Cayley octads are point sets in non-general position, although three general quadrics have eight points of intersection.

ii) Let k be any field and $X \subset \mathbf{P}^3_{k^{\mathrm{sep}}}$ be a Cayley octad that is invariant under $\mathrm{Gal}(k^{\mathrm{sep}}/k)$. Then the net Λ_X is spanned by k-rational quadrics, so that C_{Λ_X} is a plane quartic defined over k.

Proposition 2.5. *Let k be any field and $X \subset \mathbf{P}^3_{k^{\mathrm{sep}}}$ be a Cayley octad that is invariant under* $\mathrm{Gal}(k^{\mathrm{sep}}/k)$. *Assume that the quartic C_{Λ_X} is nonsingular.*

Then the operation of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ *on the 28 bitangents of C_{Λ_X} coincides with that on pairs of the eight points of X. In particular, if each of the eight points of X is defined over an extension field l of k then the 28 bitangents of C are defined over l.*

Proof. As is classically known (cf. [Sen, Proposition 5.4] or [PSV, Proposition 3.3]), each pair of points of X defines a bitangent of C_{Λ_X}, which may be rationally computed from the pair. The assertion immediately follows from this. $\square$

Proposition 2.6. *Let k be a field and $f \in k[T]$ a separable polynomial of degree* 8, *whose coefficient at T^7 vanishes. Denote by $l_0 := k[T]/(f)$ the étale algebra defined by f, and let $\alpha := (T \bmod f)$ be its natural generator.*

Furthermore, write l for the splitting field of f, and let $\sigma_1, \ldots, \sigma_8 \colon l_0 \to l$ be the eight homomorphisms, which are given by the roots of f. The operation of $\mathrm{Gal}(l/k)$ *on the homomorphisms then yields an injection $\iota\colon \mathrm{Gal}(l/k) \hookrightarrow S_8$.*

a) *Then the eight points $(1 : \sigma_i(\alpha) : \sigma_i(\alpha)^2 : \sigma_i(\alpha)^4) \in \mathbf{P}^3(l)$, for $i = 1, \ldots, 8$, form a Cayley octad X.*

b) *Moreover, X is* $\mathrm{Gal}(l/k)$*-invariant, the eight points being acted upon by* $\mathrm{Gal}(l/k)$ *as described by ι.*

c) *Suppose that, for each subset $\{i_1, \ldots, i_4\} \subset \{1, \ldots, 8\}$ of size four,*

$$\sum_{j=1}^{4} \sigma_{i_j}(\alpha) \neq 0\,.$$

Then the quartic C_{Λ_X} associated with X is nonsingular.

Proof. a) The 2-uple embedding maps $(1 : \sigma_i(\alpha) : \sigma_i(\alpha)^2 : \sigma_i(\alpha)^4)$ to

$$(1 : \sigma_i(\alpha)^2 : \sigma_i(\alpha)^4 : \sigma_i(\alpha)^8 : \sigma_i(\alpha) : \sigma_i(\alpha)^2 : \sigma_i(\alpha)^4 : \sigma_i(\alpha)^3 : \sigma_i(\alpha)^5 : \sigma_i(\alpha)^6)\,. \tag{1}$$

We first claim that the 8×10-matrix formed by the eight rows of the form (1) is of rank 7. For this, we may reorder the columns, thereby omitting the repeated ones.

The result is the 8×8-matrix

$$\begin{pmatrix} 1 & \sigma_1(\alpha) & \sigma_1(\alpha)^2 & \sigma_1(\alpha)^3 & \sigma_1(\alpha)^4 & \sigma_1(\alpha)^5 & \sigma_1(\alpha)^6 & \sigma_1(\alpha)^8 \\ 1 & \sigma_2(\alpha) & \sigma_2(\alpha)^2 & \sigma_2(\alpha)^3 & \sigma_2(\alpha)^4 & \sigma_2(\alpha)^5 & \sigma_2(\alpha)^6 & \sigma_2(\alpha)^8 \\ 1 & \sigma_3(\alpha) & \sigma_3(\alpha)^2 & \sigma_3(\alpha)^3 & \sigma_3(\alpha)^4 & \sigma_3(\alpha)^5 & \sigma_3(\alpha)^6 & \sigma_3(\alpha)^8 \\ 1 & \sigma_4(\alpha) & \sigma_4(\alpha)^2 & \sigma_4(\alpha)^3 & \sigma_4(\alpha)^4 & \sigma_4(\alpha)^5 & \sigma_4(\alpha)^6 & \sigma_4(\alpha)^8 \\ 1 & \sigma_5(\alpha) & \sigma_5(\alpha)^2 & \sigma_5(\alpha)^3 & \sigma_5(\alpha)^4 & \sigma_5(\alpha)^5 & \sigma_5(\alpha)^6 & \sigma_5(\alpha)^8 \\ 1 & \sigma_6(\alpha) & \sigma_6(\alpha)^2 & \sigma_6(\alpha)^3 & \sigma_6(\alpha)^4 & \sigma_6(\alpha)^5 & \sigma_6(\alpha)^6 & \sigma_6(\alpha)^8 \\ 1 & \sigma_7(\alpha) & \sigma_7(\alpha)^2 & \sigma_7(\alpha)^3 & \sigma_7(\alpha)^4 & \sigma_7(\alpha)^5 & \sigma_7(\alpha)^6 & \sigma_7(\alpha)^8 \\ 1 & \sigma_8(\alpha) & \sigma_8(\alpha)^2 & \sigma_8(\alpha)^3 & \sigma_8(\alpha)^4 & \sigma_8(\alpha)^5 & \sigma_8(\alpha)^6 & \sigma_8(\alpha)^8 \end{pmatrix},$$

which is clearly of rank $\leqslant 7$, as the rightmost column is a linear combination of the other seven. On the other hand, the upper left 7×7-minor is Vandermonde and thus of value

$$\prod_{1 \leqslant i < j \leqslant 7} [\sigma_i(\alpha) - \sigma_j(\alpha)] \neq 0\,,$$

which implies our claim. Consequently, the quadrics through X indeed form a net Λ_X.

In order to prove that X is indeed a Cayley octad, it now suffices to verify that the base locus of Λ_X is zero dimensional. In fact, in this case, according to Bezout, it cannot consist of more than the eight points given. To show zero dimensionality, we first observe that the net Λ_X contains the pencil L spanned by $Z(q_1)$ and $Z(q_2)$, for $q_1 := T_1^2 - T_0T_2$ and $q_2 := T_2^2 - T_0T_3$. The pencil L contains the nonsingular quadric $Q := Z(q_1 + q_2)$ and has

$$C\colon T_1^2 - T_0T_2 = T_2^2 - T_0T_3 = 0$$

as its base locus, which is an irreducible curve. Indeed, on the affine chart "$T_0 = 1$", one has the system of equations $T_1^2 = T_2$ and $T_2^2 = T_3$ that yields the rational parametrisation $\mathbf{P}^1 \to C$, $(s:t) \mapsto (s^4 : s^3t : s^2t^2 : t^4)$, for C. Consequently, the base locus of Λ_X could be of positive dimension only if the third generator of Λ_X contained the curve C entirely.

However, left exactness of the global section functor, applied to the standard exact sequences

$$0 \longrightarrow \mathscr{O}_{\mathbf{P}^3} \stackrel{\cdot(q_1+q_2)}{\longrightarrow} \mathscr{O}_{\mathbf{P}^3}(2) \longrightarrow \mathscr{O}_Q(2) \longrightarrow 0$$

and

$$0 \longrightarrow \mathscr{O}_Q \stackrel{\cdot q_1}{\longrightarrow} \mathscr{O}_Q(2) \longrightarrow \mathscr{O}_C(2) \longrightarrow 0$$

shows that no quadratic form vanishes on C, except for those linearly spanned by q_1 and q_2.

b) This assertion is clear from the construction of X.

c) Since the eight points given are clearly distinct, all that is left to verify for C_{Λ_X} being nonsingular is that no four of the points are coplanar. But this is clear, too.

Indeed, the determinants

$$\det \begin{pmatrix} 1 & \sigma_{i_1}(\alpha) & \sigma_{i_1}(\alpha)^2 & \sigma_{i_1}(\alpha)^4 \\ 1 & \sigma_{i_2}(\alpha) & \sigma_{i_2}(\alpha)^2 & \sigma_{i_2}(\alpha)^4 \\ 1 & \sigma_{i_3}(\alpha) & \sigma_{i_3}(\alpha)^2 & \sigma_{i_3}(\alpha)^4 \\ 1 & \sigma_{i_4}(\alpha) & \sigma_{i_4}(\alpha)^2 & \sigma_{i_4}(\alpha)^4 \end{pmatrix},$$

for $1 \leqslant i_1 < \ldots < i_4 \leqslant 8$, are easily calculated to be

$$\big(\sigma_{i_1}(\alpha) + \sigma_{i_2}(\alpha) + \sigma_{i_3}(\alpha) + \sigma_{i_4}(\alpha)\big) \cdot \prod_{1 \leqslant j_1 < j_2 \leqslant 4} [\sigma_{j_1}(\alpha) - \sigma_{j_2}(\alpha)]\,,$$

an expression, in which the factor on the left is nonzero by assumption while that on the right is nonzero by construction. $\square$

Remark 2.7. The more obvious choice of the eight points $(1 : \sigma_i(\alpha) : \sigma_i(\alpha)^2 : \sigma_i(\alpha)^3)$ does not lead to a Cayley octad. There is actually a net of quadrics through these eight points, spanned by $Z(T_1^2 - T_0T_2)$, $Z(T_2^2 - T_1T_3)$, and $Z(T_1T_2 - T_0T_3)$, but the base locus of this net is the twisted cubic curve.

We may thus conclude the following result.

Theorem 2.8. *Let an infinite field k of characteristic not 2, a normal and separable extension field l, and an injective group homomorphism*

$$i\colon \operatorname{Gal}(l/k) \hookrightarrow U_{36} \subset G \cong \operatorname{Sp}_6(\mathbb{F}_2) \subset S_{28}$$

be given. Then there exists a nonsingular quartic curve C over k such that l is the field of definition of the 28 bitangents and each $\sigma \in \operatorname{Gal}(l/k)$ permutes the bitangents as described by $i(\sigma) \in G \subset S_{28}$.

Proof. The natural operation of S_8 on the $28 = \frac{8\cdot 7}{2}$ 2-sets yields an injective group homomorphism $S_8 \to S_{28}$, the image of which is exactly U_{36}. Thus, the homomorphism i given induces an injection $i'\colon \operatorname{Gal}(l/k) \hookrightarrow S_8$.

This immediately yields an étale algebra l_0 of degree eight. In the case of a transitive subgroup, l_0 is the field corresponding under the Galois correspondence to the point stabiliser. In general, one has to take the direct product of the fields, corresponding to the point stabilisers of the orbits.

Moreover, by the primitive element theorem, every element of l_0 is a generator, except for the union of finitely many k-linear subspaces of lower dimension. The same is still true for the trace zero subspace

$$h := \{\alpha \in l_0 \mid \operatorname{Tr}_{l_0/k}(\alpha) = 0\} \subset l_0\,.$$

Indeed, with $\alpha \in l_0$, the element $\alpha' := \alpha - \frac{1}{8}\operatorname{Tr}_{l_0/k}(\alpha)$ of trace zero is a generator, too.

Furthermore, the eight homomorphisms $\sigma_i\colon l_0 \to l$ are k-linearly independent. Hence, the subset

$$h_c := \Big\{\alpha \in h \mid \sum_{j=1}^{4} \sigma_{i_j}(\alpha) = 0 \text{ for some } 1 \leqslant i_1 < \ldots < i_4 \leqslant 8\Big\} \subset h\,,$$

too, is a union of finitely many lower-dimensional k-linear subspaces.

Consequently, as k is infinite, there exists some element $\alpha \in h \backslash h_c$ that is a generator of the étale algebra l_0. We take $f \in k[T]$ to be the minimal polynomial of α. Then, one has $l_0 \cong k[T]/(f)$ and the generator $(T \bmod f)$ is mapped to α under this isomorphism. The coefficient of f at T^7 is zero, because of $\mathrm{Tr}_{l_0/k}(\alpha) = 0$. Moreover, f is separable, as, for α a generator, $\sigma_1(\alpha), \ldots, \sigma_8(\alpha)$ are automatically distinct. Thus, all the assumptions of Proposition 2.6 are fulfilled.

Therefore, we are given a Galois invariant Cayley octad X delivering a nonsingular plane quartic C_{Λ_X}. Proposition 2.5 shows, finally, that the Galois operation on the bitangents of C_{Λ_X} is exactly the one desired. □

Corollary 2.9 (The case that k is a number field)**.** *Let k be a number field and g a subgroup of $G \cong \mathrm{Sp}_6(\mathbb{F}_2)$ that is contained in U_{36}. Then there exists a nonsingular quartic curve C over k such that the natural permutation representation*

$$i\colon \mathrm{Gal}(\overline{k}/k) \longrightarrow G \subset S_{28}$$

on the 28 bitangents of C has the subgroup g as its image.

Proof. According to Theorem 2.8, it suffices to show that, for every number field k and each subgroup $g \subseteq U_{36}$, there exists a normal extension field l such that $\mathrm{Gal}(l/k)$ is isomorphic to g. This is a particular instance of the inverse Galois problem for number fields and thus potentially hard. But, fortunately, the groups occurring are easy enough.

In fact, 296 of the 1369 conjugacy classes of subgroups of G are contained in U_{36}. These are exactly the conjugacy classes of subgroups of $U_{36} \cong S_8$. It does not happen that two subgroups of U_{36} not being conjugate to each other become conjugate in G. Cf. Corollary A.4.

Among these 296 conjugacy classes, 268 consist of solvable groups. In this case, the inverse Galois problem has been solved by I. R. Shafarevich [Sha], cf. [NSW, Theorem 9.5.1]. The groups in the remaining 28 conjugacy classes turn out to be

i) isomorphic to a symmetric group S_n or alternating group A_n, for $n = 5$, 6, 7, or 8, in 13 cases, and

ii) isomorphic to a direct product $S_m \times S_n$, $S_m \times A_n$, or $A_m \times A_n$, in ten further cases.

iii) The remaining five cases are the simple group of order 168, occurring transitively and intransitively, the transitive subgroups of the types $\mathrm{PGL}_2(\mathbb{F}_7)$ and $\mathrm{AGL}_3(\mathbb{F}_2)$, having the orders 336 and 1344, respectively, and the subdirect product of S_5 and S_3, of order 360.

The groups of types i) and ii) are classically known to occur as Galois groups over an arbitrary number field k [MM, Theorem 9.4 and 9.5]. Moreover, the first three groups of type iii) are realised over $\mathbb{Q}(t)$, still over $\overline{\mathbb{Q}}(t)$, and consequently over $k(t)$, too, by the polynomials $f_{7,3}$, $f_{8,43}$, and $f_{8,48}$ from the tables in [MM, Appendix]. Finally, for the subdirect product, the polynomial

$$[T^3 - 3(3t + 1)T + 2(3t + 1)] \cdot [T^5 - 5(-5t + 1)T + 4(-5t + 1)] \in \mathbb{Q}(t)[T]$$

enjoys the same property. According to the Hilbert irreducibility theorem [Se, Section 9.2, Proposition 2], in each case, there are infinitely many specialisations of t to an element of k, such that the resulting polynomial over k has the same Galois group. □

Example 2.10. For the polynomial

$$f(T) := T^8 + 42T^4 + 168T^2 + 1152T + 1197 \in \mathbb{Q}[T]\,,$$

we find the nonsingular plane quartic over $\mathbb{Q}$, given by

$$\begin{aligned}2T_0^3T_2 - 5T_0^2T_2^2 - 3T_0T_1^3 - 6T_0T_1^2T_2 + 24T_0T_1T_2^2 - 8T_0T_2^3 + 6T_1^3T_2 + 12T_1^2T_2^2 \\ + 24T_1T_2^3 - 16T_2^4 = 0\,.\end{aligned}$$

Here, the Galois group of f is the simple group of order 168, realised as a transitive subgroup of S_8. This group is in fact doubly transitive, so that the operation on the 28 bitangents is transitive, too.

Example 2.11. Over the function field $\mathbb{F}_3(t)$, the polynomial

$$\begin{aligned}f(T) := T^8 + 2tT^6 + 2t^2T^5 + (t^3{+}2t^2{+}t + 2)T^4 + 2t^3T^3 + (2t^3{+}t{+}2)T^2 \\ + (t^5{+}t^4{+}t^3{+}2t^2)T + (t^6{+}t^4{+}2t^3{+}t^2{+}1) \in \mathbb{F}_3(t)[T]\end{aligned}$$

provides the same Galois group. We obtain the nonsingular plane quartic over $\mathbb{F}_3(t)$, given by

$$\begin{aligned}&(t^{13}{+}t^{12}{+}2t^{10}{+}t^8{+}2t^6{+}2t^4{+}t^3{+}t^2{+}t{+}1)T_0^4 \\ &+ (2t^9{+}2t^8{+}t^7{+}t^6{+}2t^5{+}t^4{+}2t{+}1)T_0^3T_1 \\ &+ (t^{10}{+}t^9{+}t^8{+}2t^7{+}t^6{+}2t^3{+}t^2{+}2t{+}1)T_0^3T_2 + (t^4{+}t^3{+}2t{+}1)T_0^2T_1^2 \\ &+ (t^6{+}t^5{+}2t^4{+}2t^3{+}t{+}1)T_0^2T_1T_2 + (t^7{+}2t^4{+}t^3{+}t^2{+}2t{+}2)T_0^2T_2^2 + T_0T_1^3 \\ &+ tT_0T_1^2T_2 + (t^3{+}2t^2{+}t{+}2)T_0T_1T_2^2 + (2t^3{+}t{+}2)T_0T_2^3 + 2T_1T_2^3 = 0\,.\end{aligned}$$

Example 2.12. For the polynomial

$$f(T) := T^8 - 5T^6 - T^5 + 7T^4 + T^3 + 4T^2 + 1 \in \mathbb{Q}[T]\,,$$

the nonsingular plane quartic over $\mathbb{Q}$, given by

$$\begin{aligned}T_0^4 - 2T_0^3T_1 - 5T_0^3T_2 + 6T_0^2T_1T_2 - 7T_0^2T_2^2 + 8T_0T_1^2T_2 - 6T_0T_1T_2^2 + 5T_0T_2^3 \\ + 8T_1^3T_2 + T_1^2T_2^2 - 10T_1T_2^3 {+} 2T_2^4 = 0\end{aligned}$$

results. Here, the Galois group of f is A_8. The operation on the 28 bitangents is transitive.

Example 2.13. Over the function field $\mathbb{F}_3(t)$, the polynomial

$$f(T) := T^8 + tT^5 + 2tT^2 + 1 \in \mathbb{F}_3(t)[T]$$

has Galois group A_8. We find the nonsingular plane quartic over $\mathbb{F}_3(t)$, given by

$$(t^2 + t)T_0^2T_1^2 + T_0T_1^3 + (t^2 + t)T_0^3T_2 + T_0^2T_1T_2 + 2tT_0T_2^3 + 2T_1T_2^3 = 0\,.$$

Remark 2.14. The examples above were chosen from the huge supply in the hope that they are of some particular interest. For instance, in both cases the groups occurring as Galois groups are simple. The corresponding degree two del Pezzo surfaces are, independently of the twist chosen, all of Picard rank 1.

Remark 2.15 (The situation of an algebraically closed base field—Moduli schemes). There is a coarse moduli scheme of nonsingular quartic curves, provided by Geometric Invariant Theory. It is nothing but the quotient $\mathscr{M} := \mathscr{V}/\mathrm{PGL}_3$, for $\mathscr{V} \subset \mathbf{P}(\mathrm{Sym}^4(k^3)^*) \cong \mathbf{P}^{14}$ the open subscheme parametrising nonsingular plane quartics. Observe here that, by [Mu, Remark 1.13], every nonsingular plane quartic corresponds to a PGL_3-stable point on $\mathbf{P}(\mathrm{Sym}^4(k^3)^*)$.

The PGL_3-invariants have been determined by J. Dixmier [Di] and T. Ohno, Ohno in his unpublished paper [Oh] proving completeness. There is a more recent treatment due to A.-S. Elsenhans [El], who also provided an implementation into `magma`. The Dixmier–Ohno invariants yield an embedding

$$\mathscr{V}/\mathrm{PGL}_3 \cong \mathscr{M} \hookrightarrow \mathbf{P}(1,2,3,3,4,4,5,5,6,6,7,7,9)$$

into a weighted projective space.

On the other hand, there is the moduli space $\mathscr{M}^{\mathrm{ev}}$ of projective equivalence classes of Cayley octads, cf. [Do, Corollary 6.3.12]. As is classically known, every nonsingular plane quartic may be obtained from a Cayley octad in 36 non-equivalent ways [He, §14]. Accordingly, $\mathscr{M}^{\mathrm{ev}}$ naturally provides a 36 : 1 étale covering

$$\pi\colon \mathscr{M}^{\mathrm{ev}} \longrightarrow \mathscr{M}$$

of the moduli scheme $\mathscr{M}$ of nonsingular plane quartics [Do, Theorem 6.3.19].

Our construction, from the geometric point of view, describes a particular kind of Cayley octads, namely those of the type

$$\{(1\!:\!x_i\!:\!x_i^2\!:\!x_i^4) \in \mathbf{P}^3(k) \mid i = 1,\ldots,8\}\,,$$

for $\{x_1,\ldots,x_8\} \subset k$ a subset consisting of eight elements such that $x_1+\cdots+x_8 = 0$. It therefore yields a rational map

$$\iota\colon \mathbf{A}^7 \dashrightarrow \mathscr{M}^{\mathrm{ev}}\,.$$

Moreover, a simple experiment in `magma` reveals the fact that ι is dominant. Indeed, we verified dominance of the composition $\pi\circ\iota\colon \mathbf{A}^7 \dashrightarrow \mathscr{M}$ by showing that the tangent map $T\pi_x\colon T_x\mathbf{A}^7 \to T_{\iota(x)}\mathscr{M}$ at a single point $x \in \mathbf{A}^7$ is surjective. Here, we used the first author's code [El] for calculating the Dixmier–Ohno invariants of a ternary quartic form.

As a consequence, we find that both, $\mathscr{M}^{\mathrm{ev}}$ and $\mathscr{M}$ are unirational varieties. This result, however, is not new. In fact, $\mathscr{M}^{\mathrm{ev}}$ and $\mathscr{M}$ are even known to be rational. For $\mathscr{M}$, this is due to P. Katsylo [Ka, Theorem 0.1], while the rationality of $\mathscr{M}^{\mathrm{ev}}$ follows from this in view of [Do, Theorem 6.3.19].

Anyway, our approach to construct quartic curves with an arbitrarily given Galois operation on the 28 bitangents, having the image of Galois contained in

$U_{36} \subset G \cong \mathrm{Sp}_6(\mathbb{F}_2)$, seemed to run quite smoothly. It is our distinct impression that the rationality of the moduli scheme $\mathscr{M}^{\mathrm{ev}}$ is the proper reason for this.

3. Application to del Pezzo surfaces of degree two–Twisting

There is the double cover $p\colon \widetilde{G} \to G$ of finite groups, for $W(E_7) \cong \widetilde{G} \subset S_{56}$ and $\mathrm{Sp}_6(\mathbb{F}_2) \cong G \subset S_{28}$, which is given by the operation on the size two blocks. The kernel of p is exactly the centre $Z \subset \widetilde{G}$. For a subgroup $H \subset \widetilde{G}$, one therefore has two options.

i) Either $p|_H\colon H \to p(H)$ is two-to-one. Then $H = p^{-1}(h)$, for $h := p(H)$. In this case, H contains the centre of $\widetilde{G}$ and, as abstract groups, one has an isomorphism $H \cong p(H) \times \mathbb{Z}/2\mathbb{Z}$.

ii) Or $p|_H\colon H \to p(H)$ is bijective.

In our geometric setting, the first case is the generic one. More precisely, let $C\colon q = 0$ be a nonsingular plane quartic such that the 28 bitangents are acted upon by the group $h \subseteq G$. Then, for λ an indeterminate, the 56 exceptional curves on $S_\lambda\colon \lambda w^2 = q$ are operated upon by $h \times \mathbb{Z}/2\mathbb{Z}$. For particular choices of λ, every subgroup of $\widetilde{G}$ may be realised that has image h under the projection p.

Lemma 3.1. *Let a field k of characteristic not 2, a normal and separable extension field l, and an injective group homomorphism $i\colon \mathrm{Gal}(l/k) \hookrightarrow \widetilde{G}$ be given. Write l' for the subfield corresponding to $i^{-1}(Z)$ under the Galois correspondence.*

a) *Then there is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Gal}(l/k) & \stackrel{i}{\hookrightarrow} & \widetilde{G} \\ {\scriptstyle \mathrm{res}}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{Gal}(l'/k) & \stackrel{\bar{\imath}}{\hookrightarrow} & G\,, \end{array}$$

the downward arrow on the left being the restriction.

b) *Let $C\colon q = 0$ be a nonsingular plane quartic over k, the 28 bitangents of which are defined over l' and acted upon by $\mathrm{Gal}(l'/k)$ as described by $\bar{\imath}$. Then there exists some $\lambda \in k^*$ such that the 56 exceptional curves of the degree two del Pezzo surface*

$$S_\lambda\colon \lambda w^2 = q$$

are defined over l and each automorphism $\sigma \in \mathrm{Gal}(l/k)$ permutes them as described by $i(\sigma) \in \widetilde{G} \subset S_{56}$.

Proof. Cf. [EJ17, Theorem 4.2]. $\square$

Thus, from our main result on plane quartics, we may draw the following conclusion.

Theorem 3.2. *Let an infinite field k of characteristic not 2, a normal and separable extension field l, and an injective group homomorphism*

$$i\colon\ \mathrm{Gal}(l/k) \hookrightarrow p^{-1}(U_{36})$$

be given. Then there exists a degree two del Pezzo surface S over k such that l is the field of definition of the 56 exceptional curves and each $\sigma \in \mathrm{Gal}(l/k)$ permutes them as described by $i(\sigma) \in \widetilde{G} \subset S_{56}$.

Proof. This follows from Lemma 3.1 together with Theorem 2.8. □

Remark 3.3 (2-torsion Brauer classes). There is a unique subgroup

$$H_{\mathrm{Br}} \subset p^{-1}(U_{36}) \subset \widetilde{G} \subset S_{56}$$

of index 2 that is intransitive of orbit type $[28, 28]$. Clearly, one has $H_{\mathrm{Br}} \cong U_{36} \cong S_8$. The subgroup H_{Br} is of index 72 in $\widetilde{G}$.

Assume that S is a degree two del Pezzo surface over a field k such that the Galois group operating on the 56 exceptional curves is a subgroup of H_{Br}. In this case, S carries a global Brauer class $\alpha \in \mathrm{Br}(S_\lambda)_2$. In fact, the 2-torsion Brauer classes on degree two del Pezzo surfaces have been systematically studied by P. Corn in [Co]. It turns out that the conjugacy classes of subgroups of $W(E_7)$ that lead to such a Brauer class form a partially ordered set with exactly two maximal elements. The subgroup H_{Br} described above is one of them. It yields the Brauer classes of the second type in Corn's terminology.

There can be no doubt that this type is more interesting than the other. A short experiment shows the following facts.

i) For 176 out of the 296 conjugacy classes H of subgroups of H_{Br}, the Brauer class α is nontrivial.

ii) For 87 of these, the Brauer class α is non-cyclic. I.e., there is no normal subgroup $H' \subset H$ with cyclic quotient annihilating the class. Moreover, somewhat surprisingly, in each of these 87 cases, the restriction to the 2-Sylow subgroup of H is still non-cyclic.

This shows that, in the situation that k is a number field and ν one of its places, the evaluation of such a Brauer class at a k_ν-rational point cannot be computed in the usual, class field theoretic way (cf., e.g., [EJ10, §6]).

Remark 3.4. Again, two subgroups $U_1, U_2 \subseteq H_{\mathrm{Br}}$ not being conjugate to each other do not become conjugate when considered as subgroups of $\widetilde{G}$.

Proof. Indeed, observe the commutative diagram below,

$$\begin{tikzcd} H_{\mathrm{Br}} \arrow[r, hook] \arrow[d, "p|_{H_{\mathrm{Br}}}"', "\cong"] & \widetilde{G} \arrow[d, two heads, "p"] \\ U_{36} \arrow[r, hook] & G\,. \end{tikzcd}$$

From this, one sees that if $U_1, U_2 \subseteq H_{\mathrm{Br}}$ are conjugate in $\widetilde{G}$ then $p(U_1), p(U_2) \subseteq U_{36}$ are conjugate in G. But then Corollary A.4 shows that $p(U_1), p(U_2) \subseteq U_{36}$ are already conjugate in U_{36}. And, finally, since $p|_{H_{\mathrm{Br}}}$ is an isomorphism, U_1, U_2 must be conjugate to each other in H_{Br}. □

4. Another application: Cubic surfaces with a Galois invariant double-six

A nonsingular cubic surface over an algebraically closed field contains exactly 27 lines. The maximal subgroup $G_{\max} \subset S_{27}$ that respects the intersection pairing is isomorphic to the Weyl group $W(E_6)$ [Ma, Theorem 23.9] of order 51 840.

A double-six (cf. [Ha, Remark V.4.9.1] or [Do, Subsection 9.1.1]) is a configuration of twelve lines $E_1, \ldots, E_6, E'_1, \ldots, E'_6$ such that

i) $E_1, \ldots, E_6$ are mutually skew,

ii) $E'_1, \ldots, E'_6$ are mutually skew, and

iii) $E_i \cdot E'_j = 1$, for $i \neq j$, $1 \leqslant i, j \leqslant 6$, and $E_i \cdot E'_i = 0$ for $i = 1, \ldots, 6$.

Every cubic surface contains exactly 36 double-sixes, which are transitively acted upon by $G_{\max}$ [Do, Theorem 9.1.3]. Thus, there is an index-36 subgroup $U_{\mathrm{ds}} \subset G_{\max}$ stabilising a double-six. This is one of the maximal subgroups of $G_{\max} \cong W(E_6)$. Up to conjugation, $W(E_6)$ has maximal subgroups of indices 2, 27, 36, 40, 40, and 45.

As an application of Theorem 2.8 and Lemma 3.1, we now have a simpler proof for the following result, which was shown in [EJ17, Theorem 5.1].

Theorem 4.1. *Let an infinite field k of characteristic not 2, a normal and separable extension field l, and an injective group homomorphism*

$$i\colon \operatorname{Gal}(l/k) \hookrightarrow U_{\mathrm{ds}} \subset G_{\max} \cong W(E_6) \subset S_{27}$$

be given. Then there exists a nonsingular cubic surface S over k such that

i) *the 27 lines on S are defined over l and each $\sigma \in \operatorname{Gal}(l/k)$ permutes them as described by $i(\sigma) \in U_{\mathrm{ds}} \subset S_{27}$.*

ii) *S is k-unirational.*

Proof. There is an injective homomorphism $\iota\colon W(E_6) \hookrightarrow W(E_7)$ that corresponds to the blow-up of a point on a cubic surface. The subgroup $\iota(U_{\mathrm{ds}}) \subset W(E_7)$ is of index $36 \cdot 56 = 2016$. It is sufficient to show that the image $\overline{\iota}(U_{\mathrm{ds}})$ under the composition

$$\overline{\iota}\colon W(E_6) \overset{\iota}{\hookrightarrow} W(E_7) \overset{p}{\twoheadrightarrow} W(E_7)/Z \cong G$$

is contained in U_{36}. Indeed, then Theorem 3.2 yields a degree two del Pezzo surface S' of degree two with a k-rational line L and the proposed Galois operation on the 27 lines that do not meet L. Blowing down L, we obtain a nonsingular cubic surface S satisfying i). As there is a k-rational point on S, the blow down of L, S is k-unirational [Ko02, Theorem 2].

To show the group-theoretic claim, we first observe that the homomorphism $\overline{\iota}$ is still an injection. Furthermore, the subgroup $\overline{\iota}(W(E_6)) \subset G$ of index 28 is exactly the point stabiliser of the transitive group $G \subset S_{28}$. The subgroup $U_{36} \subset S_{28}$ is still transitive, as $U_{36} \cong S_8$, operating on 2-sets, and S_8 is doubly transitive. Thus, $U_{36} \cap \overline{\iota}(W(E_6))$ is of index 28 in U_{36}. Moreover, $\overline{\iota}$ yields an isomorphism

$$\overline{\iota}|_{\overline{\iota}^{-1}(U_{36})}\colon \overline{\iota}^{-1}(U_{36}) \xrightarrow{\cong} U_{36} \cap \overline{\iota}(W(E_6))\,.$$

In particular, $\overline{\iota}^{-1}(U_{36})$ is of the same order as $U_{36} \cap \overline{\iota}(W(E_6))$ and hence of index 36 in the group $W(E_6)$. Therefore, up to conjugation, $\overline{\iota}^{-1}(U_{36}) = U_{\mathrm{ds}}$, which implies $\overline{\iota}(U_{\mathrm{ds}}) = \overline{\iota}(\overline{\iota}^{-1}(U_{36})) \subseteq U_{36}$, as required. $\square$

Appendix A. A group-theoretic observation

Proposition A.1. *Let G be a finite group and $H \subset G$ a subgroup. Consider the permutation representation*

$$\begin{aligned} \iota\colon G &\longrightarrow \mathrm{Sym}(G/H)\,, \\ g &\mapsto (aH \mapsto gaH) \end{aligned}$$

on left cosets modulo H, and assume that, for all $\omega, \omega' \in G/H$, $\omega' \neq \omega$, the following two conditions are satisfied.

i) *There is some $g \in G$ such that $\iota(g) \in \mathrm{Sym}(G/H)$ interchanges ω and ω'. I.e., such that the permutation $\iota(g)$ contains the 2-cycle $(\omega\omega')$ in its canonical decomposition into disjoint cycles.*

ii) *The twofold point stabiliser $(G_\omega)_{\omega'}$ is a group having only inner automorphisms.*

Then the following is true. If two subgroups $U_1, U_2 \subseteq H$ are conjugate in G then they are conjugate already in H.

Proof. Consider, at first, two elements $\omega, \omega' \in G/H$ such that $\omega' \neq \omega$. Then, according to assumption i), there is some $g \in G$ such that $\iota(g) = (\omega\omega') \cdots$. Conjugation by g then clearly provides an automorphism of $(G_\omega)_{\omega'}$. Moreover, by assumption ii), that must be inner. I.e., there exists some $h \in (G_\omega)_{\omega'}$ satisfying $hxh^{-1} = gxg^{-1}$ for every $x \in (G_\omega)_{\omega'}$, and therefore $xh^{-1}g = h^{-1}gx$. In other words, we found an element

$$c_{\omega,\omega'} := h^{-1}g \in C_G\big((G_\omega)_{\omega'}\big)$$

in the centraliser of $(G_\omega)_{\omega'}$ that is non-trivial, because of $i(c_{\omega,\omega'}) = (\omega\omega') \cdots$.

Now suppose that

$$uU_1u^{-1} = U_2 \tag{2}$$

for some $u \in G$. We let $\omega := H$, i.e. the trivial coset. Because of $U_1, U_2 \subseteq H$, ω is a fixed point of U_1, as well as of U_2. Moreover, assumption (2) shows that $\omega' := u^{-1}(\omega)$ must be a fixed point of U_1, too.

If $\omega' = \omega$ then $u \in H$ and the assertion is trivially true. Otherwise, $U_1 \subseteq (G_\omega)_{\omega'}$ and one has

$$\begin{aligned} U_2 &= uU_1u^{-1} \\ &= uc_{\omega,\omega'}U_1c_{\omega,\omega'}^{-1}u^{-1}\,, \end{aligned}$$

since $c_{\omega,\omega'}$ is a central element. Here, $uc_{\omega,\omega'}$ fixes ω, which shows that $uc_{\omega,\omega'} \in H$. $\square$

Remarks A.2. i) The permutation representation ι is always transitive. Thus, in the assumptions i) and ii), it suffices to restrict considerations to the case that $\omega := H$.

ii) It is, of course, possible to find a purely group-theoretic formulation, avoiding permutations. For instance, the twofold point stabiliser of $\omega = H$ and $\omega' = xH$, for $x \notin H$, is $H \cap xHx^{-1}$.

Remark A.3. A relevant particular case occurs when ι doubly transitive. Indeed, in this case, assumption i) is automatically fulfilled. Furthermore, all twofold point stabilisers $(G_\omega)_{\omega'}$ are isomorphic to each other.

Corollary A.4. *If two subgroups $U_1, U_2 \subseteq U_{36}$ are conjugate in $G \cong \mathrm{Sp}_6(\mathbb{F}_2)$ then they are conjugate already in U_{36}.*

Proof. In our case, $G \cong \mathrm{Sp}_6(\mathbb{F}_2)$ and $H = U_{36} \cong S_8$. As this is a maximal subgroup, the permutation representation ι is primitive of degree 36. It is well-known that $\mathrm{Sp}_6(\mathbb{F}_2)$ allows only one primitive permutation representation in this degree, and that this is doubly transitive [DM, Theorem 7.7.A, for $n = 3$].

Moreover, $G_\omega \cong S_8$ allows only one conjugacy class of subgroups of index 35, and that contains the imprimitive wreath product $S_4 \wr S_2$. Thus, all the twofold point stabilisers of ι are isomorphic to $S_4 \wr S_2$. Finally, this group allows only one faithful permutation representation in degree 8 and, hence, has only inner automorphisms. Thus, Proposition A.1 applies and yields the assertion. □

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