ON PLANE QUARTICS WITH A GALOIS INVARIANT STEINER HEXAD

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Abstract. We describe a construction of plane quartics with prescribed Galois operation on the 28 bitangents, in the particular case of a Galois invariant Steiner hexad. As an application, we solve the inverse Galois problem for degree two del Pezzo surfaces in the corresponding particular case.

1. Introduction

It is well known [Pl] that a nonsingular plane quartic curve C over an algebraically closed field of characteristic $\neq 2$ has exactly 28 bitangents. The same is still true if the base field is only separably closed, as is easily deduced from [Va, Theorem 1.6]. If C is defined over a separably non-closed field k then the bitangents are defined over a finite extension field l of k, which is normal and separable, and permuted by the Galois group $\mathrm{Gal}(l/k)$.

By far not every permutation may occur. In order to illustrate this, let us write Ω_C for the set of all bitangents of $C_{\overline{k}}$. First of all, every pair $\{L, L'\} \in \Omega_C^{(2)}$ of bitangents defines in a natural way a divisor class

$$\Pi(\{L, L'\}) \in \mathrm{Pic}(C_{k^{\mathrm{sep}}})_2 = \mathrm{Pic}(C_{\overline{k}})_2 \cong H^1_{\text{ét}}(C_{\overline{k}}, \mathbb{Z}/2\mathbb{Z}) .$$

The mapping $\Pi\colon \Omega_C^{(2)} \to \mathrm{Pic}(C_{\overline{k}})_2$ is exactly six-to-one onto $\mathrm{Pic}(C_{\overline{k}})_2 \backslash \{\mathscr{O}_{\overline{k}}\}$, the preimage of an element being classically called a Steiner hexad. This shows that a permutation $\sigma \in \mathrm{Sym}\,\Omega_C \cong S_{28}$ can be admissible only if the induced operation $\sigma^{(2)} \in \mathrm{Sym}\,\Omega_C^{(2)} \cong S_{28\cdot 27/2} = S_{378}$ on 2-sets keeps the 63 Steiner hexads as a block system.

Moreover, $\mathrm{Pic}(C_{\overline{k}})_2$ is canonically equipped with the Weil pairing

$$\langle .\,,.\rangle\colon \mathrm{Pic}(C_{\overline{k}})_2 \times \mathrm{Pic}(C_{\overline{k}})_2 \longrightarrow \mu_2 .$$

An admissible permutation therefore must provide an automorphism of $\mathrm{Pic}(C_{\overline{k}})_2$ that is symplectic with respect to $\langle .\,,.\rangle$.

It turns out that the subgroups $G_C \subset \mathrm{Sym}\,\Omega_C$ of all permutations that are admissible in the sense described are, independently of the choice of C, permutation isomorphic to one and the same subgroup $G \subset S_{28}$. A natural question arising is thus the following.

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Question. Given a field k and a subgroup $g \subseteq G$, does there exist a nonsingular plane quartic C over k, for which the group homomorphism $\mathrm{Gal}(k^{\mathrm{sep}}/k) \to G \subset S_{28}$, given by the Galois operation on the 28 bitangents, has the subgroup g as its image?

Remark 1.1. One has that $G \cong \mathrm{Sp}_6(\mathbb{F}_2)$. This is the simple group of order 1 451 520. Up to conjugation, it has only one subgroup of index 28, which is a maximal subgroup. Hence, the permutation group $G \subset S_{28}$ is primitive. It is even 2-transitive in view of [DM, Theorem 7.7A].

On the other hand, as a permutation group, $G \subset S_{28}$ is self-normalising. Indeed, the centraliser $C_{S_{28}}(G)$ is trivial by [DM, Theorem 4.2A.(vi)] and $\mathrm{Sp}_6(\mathbb{F}_2)$ is known to have only inner automorphisms. This shows that the permutation isomorphisms $G_C \stackrel{\cong}{\to} G$ are uniquely determined up to conjugation by elements of G. In particular, the question above depends only on the conjugacy class of the subgroup $g \subseteq G$.

The group $G \cong \mathrm{Sp}_6(\mathbb{F}_2)$ has 1369 conjugacy classes of subgroups. Among these, there are eight maximal subgroups, which are of indices 28, 36, 63, 120, 135, 315, 336, and 960, respectively.

An example of a nonsingular plane quartic over $\mathbb{Q}$ such that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ operates on the bitangents via the full $\mathrm{Sp}_6(\mathbb{F}_2)$ has been constructed by T. Shioda [Shi, Section 3] in 1993 and, almost at the same time, by R. Erné [Er, Corollary 3]. Moreover, there is an obvious approach to construct examples for the groups contained in the index 28 subgroup. Indeed, in this case, there is a rational bitangent. One may start with a cubic surface with the right Galois operation [EJ15], blow-up a rational point, and use the connection between degree two del Pezzo surfaces and plane quartics [Ko96, Theorem 3.3.5], cf. the application discussed below.

In this article, we deal with the subgroup $U_{63} \subset G$ of index 63 and the groups contained within. More precisely, we show the following result, which answers, in the case that the base field is infinite and of characteristic $\neq 2$, a more refined question than the one asked above.

Theorem 1.2. *Let an infinite field k of characteristic not 2, a normal and separable extension field l, and an injective group homomorphism*

$$i\colon \mathrm{Gal}(l/k) \hookrightarrow U_{63}$$

be given. Then there exists a nonsingular quartic curve C over k such that l is the field of definition of the 28 bitangents and each $\sigma \in \mathrm{Gal}(l/k)$ permutes the bitangents as described by $i(\sigma) \in G \subset S_{28}$.

Among the 1369 conjugacy classes of subgroups of $G \cong \mathrm{Sp}_6(\mathbb{F}_2)$, 1155 are contained in U_{63}. By which we mean that they have a member that is contained in a fixed subgroup of index 63.

The maximal subgroup U_{63} has a geometric meaning. Namely, $G \cong \mathrm{Sp}_6(\mathbb{F}_2)$ operates transitively on the 63 elements of $\mathrm{Pic}(C_{\overline{k}})_2 \backslash \{\mathscr{O}_{C_{\overline{k}}}\}$. Thus, the subgroup $U_{63} \subset G$ is just the point stabiliser and the inclusion $\mathrm{Gal}(l/k) \hookrightarrow U_{63}$ expresses the fact that there is a k-rational divisor class in $\mathrm{Pic}(C)_2$. Consequently, there is a Galois invariant Steiner hexad on C, too.

Furthermore, in this case, one of the corresponding del Pezzo surfaces of degree two has a k-rational conic bundle. In fact, there are two such bundles, which are mapped to each other under the Geiser involution. We use conic bundles for our proof of existence, which is completely constructive.

Remarks 1.3. i) As noticed above, a quartic C that provides a solution for a homomorphism $i\colon \operatorname{Gal}(l/k) \hookrightarrow G$ may, as well, serve as a solution for any homomorphism $\varphi_g \circ i$ differing from i by an inner automorphism φ_g of G, for some $g \in G$.

In Theorem 1.2, we require $\operatorname{im} i$ to be contained in the self-normalising subgroup $U_{63} \subset G$. Thus, in general, one may expect that $\varphi_g \circ i$ maps to U_{63} only for $g \in U_{63}$. I.e., we may disturb i by the inner automorphisms of U_{63}.

ii) However, our approach is based on conic bundles with six split fibres. It provides, a priori, a quartic with the right Galois operation, up to an inner automorphism of $S_2 \wr S_6$. As, however, $U_{63} \cong (S_2 \wr S_6) \cap A_{12} \subsetneqq S_2 \wr S_6$, a rather sophisticated monodromy argument is necessary in order to complete the proof for the main result.

As an application, one may answer the analogous question for degree two del Pezzo surfaces. The double cover of $\mathbf{P}^2$, ramified at a nonsingular quartic curve C, is a del Pezzo surface of degree two. Here, considerations can be made that are very similar to the ones above. First of all, it is well known [Ma, Theorem 26.2.(iii)] that a del Pezzo surface S of degree two over an algebraically closed field contains exactly 56 exceptional curves, i.e. such of self-intersection number (-1). Again, the same is true when the base field is only separably closed. If S is defined over a separably non-closed field k then the exceptional curves are defined over a normal and separable finite extension field l of k and permuted by $\operatorname{Gal}(l/k)$. Once again, not every permutation may occur. The maximal subgroup $\widetilde{G} \subset \operatorname{Sym} \widetilde{\Omega}_C \cong S_{56}$ that respects the intersection pairing is isomorphic to the Weil group $W(E_7)$ [Ma, Theorem 23.9].

Every bitangent of C is covered by exactly two of the exceptional curves of S. Thus, for the operation of $\operatorname{Gal}(k^{\mathrm{sep}}/k)$ on the 56 exceptional curves on S, there seem to be two independent conditions. On one hand, $\operatorname{Gal}(k^{\mathrm{sep}}/k)$ must operate via a subgroup of $W(E_7) \cong \widetilde{G} \subset \operatorname{Sym} \widetilde{\Omega}_C$. On the other hand, the induced operation on the blocks of size two must take place via a subgroup of $\operatorname{Sp}_6(\mathbb{F}_2) \cong G \subset \operatorname{Sym} \Omega_C \cong S_{28}$. It turns out, however, that there is an isomorphism $W(E_7)/Z \xrightarrow{\cong} \operatorname{Sp}_6(\mathbb{F}_2)$, for $Z \subset W(E_7)$ the centre, that makes the two conditions equivalent.

The group $\widetilde{G} \cong W(E_7)$ already has 8074 conjugacy classes of subgroups. Two subgroups with the same image under the quotient map $p\colon \widetilde{G} \twoheadrightarrow \widetilde{G}/Z \xrightarrow{\cong} G$ correspond to del Pezzo surfaces of degree two that are quadratic twists of each other. Theorem 1.2 therefore extends word-by-word to del Pezzo surfaces of degree two and homomorphisms $\operatorname{Gal}(l/k) \hookrightarrow \widetilde{G}$ with image contained in $p^{-1}(U_{63})$.

There are further applications of Theorem 1.2 that concern cubic surfaces. For instance, we refine our previous result [EJ10] on the existence of cubic surfaces with a Galois invariant double six and generalise it from $\mathbb{Q}$ to an arbitrary infinite field of characteristic not 2.

Organisation of the article. Section 2 summarises several general results on plane quartics, degree 2 del Pezzo surfaces, Steiner hexads, and conic bundles, which are necessary for our arguments. They are without doubt well-known to experts. Thus, concerning this part, we do not claim any originality, except for the presentation. Section 3 then describes our approach to the construction of plane quartics in detail, thereby proving Theorem 1.2. The explicit description of a conic bundle with six split fibres that are acted upon by Galois in a prescribed way, given in Proposition 3.5, is the heart of this approach. We deduce the main result in Section 4 by providing the necessary monodromy argument, which is related to the fact that $U_{63} \cong (S_2 \wr S_6) \cap A_{12} \subsetneqq S_2 \wr S_6$. Degree two del Pezzo surfaces and their quadratic twists of are discussed afterwards, in Section 5, and, finally, we present the applications to cubic surfaces of particular types in Sections 6 and 7. All calculations are with `magma` [BCP].

Remarks 1.4. i) Theorem 1.2 is clearly not true, in general, when k is a finite field. For example, there cannot be a nonsingular quartic curve over $\mathbb{F}_3$, all whose bitangents are $\mathbb{F}_3$-rational, simply because the projective plane contains only 13 $\mathbb{F}_3$-rational lines.

ii) We ignore about characteristic 2 in this article, as this case happens to be very different. Even over an algebraically closed field, a plane quartic cannot have more than seven bitangents [SV, p. 60].

Conventions and notations. In this article, we follow standard conventions and notations from Algebra and Algebraic Geometry, except for the following.

i) By a *field,* we mean a field of characteristic $\neq 2$. For the convenience of the reader, the assumption on the characteristic will be repeated in the formulations of our final results, but not during the intermediate steps.

ii) When V is a finite-dimensional vector space over a field k, then we denote its associated affine space $\operatorname{Spec}\operatorname{Sym} V^\vee$ in the category of schemes by $\mathbf{A}(V)$. The elements of V are then naturally in bijection with the k-rational points on $\mathbf{A}(V)$.

When E is an affine space, acted upon transitively and freely by the vector space V, we put $\mathbf{A}(E) := \operatorname{Spec}\operatorname{Pol}(E)$, for $\operatorname{Pol}(E)$ the algebra of polynomial functions on E. This is simply the pull-back of the k-algebra $\operatorname{Sym} V^\vee$ under any of the bijections $\iota_{v_0}\colon E \to V$, $v_0 + v \mapsto v$.

iii) For a field k and an integer $d > 0$, we write $k[T]_d$ for the set of all monic polynomials of degree d with coefficients in k. This is an affine space under the d-dimensional k-vector space of the polynomials of degree $< d$.

2. Generalities on plane quartics and degree two del Pezzo surfaces

Definition 2.1. Let $C \subset \mathbf{P}^2_k$ be a plane curve over a field k.

i) Then, by a *contact conic* of C, one means a conic $D \subset \mathbf{P}^2_k$ such that, at every geometric point $p \in (C \cap D)(\overline{k})$ of intersection, the intersection multiplicity is even.

ii) With a contact conic D, one associates an invertible sheaf $\mathscr{D} \in \mathrm{Pic}(C)_2$, i.e. one that is annihilated by 2 in the Picard group. Just put

$$\mathscr{D} := \mathscr{O}(\tfrac{1}{2}C.D) \otimes \mathscr{O}_{\mathbf{P}^2}(-1)|_C\,,$$

for $C.D$ the intersection divisor of D with C.

Let C be a plane quartic over an algebraically closed field. Then the genus of C is $g = \frac{3 \cdot 2}{2} = 3$, which implies that $\mathrm{Pic}(C)_2$ is a 6-dimensional vector space over $\mathbb{F}_2$. In this situation, the union $L \cup L'$ of two bitangents $L \neq L'$ is a degenerate contact conic. If L and L' touch C in p_1 and p_2 and p'_1 and p'_2, respectively, then the associated invertible sheaf in $\mathrm{Pic}(C)_2$ is

$$\begin{aligned}\mathscr{O}_C(\tfrac{1}{2}C.L) \otimes \mathscr{O}_C(\tfrac{1}{2}C.L') \otimes \mathscr{O}_{\mathbf{P}^2}(-1)|_C &= \mathscr{O}_C(\tfrac{1}{2}C.L - \tfrac{1}{2}C.L') \\ &= \mathscr{O}_C((p_1) + (p_2) - (p'_1) - (p'_2)) \\ &= \mathscr{O}_C((p'_1) + (p'_2) - (p_1) - (p_2))\,.\end{aligned}$$

This invertible sheaf is automatically nontrivial. Indeed, otherwise $\mathscr{O}_C((p_1) + (p_2))$ would have a non-constant section. However, one has $\mathscr{O}_C(2(p_1) + 2(p_2)) \cong \mathscr{O}_{\mathbf{P}^2}(1)|_C$ and the fact that $H^1(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}(-3)) = 0$ ensures that the latter sheaf has no global sections other than restrictions of global linear forms. Hence,

$$\Gamma(C, \mathscr{O}_C(2(p_1) + 2(p_2))) = \langle \tfrac{T_0}{l}, \tfrac{T_1}{l}, \tfrac{T_2}{l} \rangle\,, \tag{1}$$

for T_0, T_1, and T_2 the coordinate functions on $\mathbf{P}^2$ and l the linear form defining L. Since, as a rational function on C, none of the linear combinations has only simple poles, one sees that

$$h^0(C, \mathscr{O}_C((p_1) + (p_2))) = 1\,. \tag{2}$$

Remark 2.2. As the conclusion of the considerations just made, one obtains a canonical mapping

$$\begin{aligned}\Pi\colon \Omega_C^{(2)} = \{\text{pairs of bitangents of } C\} &\longrightarrow \mathrm{Pic}(C)_2 \backslash \{\mathscr{O}_C\}\,, \\ \{L, L'\} &\mapsto \mathscr{O}_C(\tfrac{1}{2}C.L - \tfrac{1}{2}C.L')\end{aligned}$$

from the $\frac{28 \cdot 27}{2} = 378$ pairs of bitangents of C to the 63 nonzero elements in $\mathrm{Pic}(C)_2$.

Proposition 2.3. *Let $C \subset \mathbf{P}^2_k$ be a nonsingular plane quartic over a field k. Assume that there is given an invertible sheaf $\mathscr{L} \in \mathrm{Pic}(C)_2 \backslash \{\mathscr{O}_C\}$, i.e. one that is nontrivial, 2-torsion in the Picard group, and defined over k.*

a) *Then C may be written in the symmetric determinantal form*

$$C\colon q^2 - q_1 q_2 = 0\,,$$

for q, q_1, and q_2 three quadratic forms with coefficients in k.

b) *Furthermore, the equations $q_1 = 0$ and $q_2 = 0$ define contact conics of C that are associated with $\mathscr{L}$.*

c) *All contact conics associated with $\mathscr{L}$ form a one-dimensional family. They are of the type $K_{(s:t)}\colon s^2 q_1 + 2stq + t^2 q_2 = 0$, for $(s:t) \in \mathbf{P}^1$.*

d) *The double cover* $S\colon w^2 = q^2 - q_1q_2$ *carries two* k*-rational conic bundles, the projections of which down to* $\mathbf{P}^2_k$ *coincide with the family* $K_{(s:t)}$.

Proof. a) and b) Since C is a plane quartic and nonsingular, the adjunction formula shows that $\mathscr{O}_{\mathbf{P}^2}(1)|_C = \mathscr{K}_C$ is the canonical sheaf. Clearly, one has $\deg \mathscr{O}_{\mathbf{P}^2}(1)|_C = 4$. Moreover, $\mathscr{L}^\vee$ is an invertible sheaf that is nontrivial and of degree 0, so that it has no non-zero section. Therefore, the Theorem of Riemann-Roch shows that

$$h^0(C, \mathscr{L}\otimes\mathscr{O}_{\mathbf{P}^2}(1)|_C) = h^0(C, \mathscr{L}\otimes\mathscr{K}_C) - h^0(C, \mathscr{L}^\vee) = 4 + 1 - g = 2\,.$$

I.e., there is a pencil of effective divisors defining the invertible sheaf $\mathscr{L}\otimes\mathscr{O}_{\mathbf{P}^2}(1)|_C$. Let $(p_1)+\cdots+(p_4)$ be such an effective divisor for $\mathscr{L}\otimes\mathscr{O}_{\mathbf{P}^2}(1)|_C$ and $(p_5)+\cdots+(p_8)$ be another. Then

$$2(p_1) + \cdots + 2(p_4)\,, \quad 2(p_5) + \cdots + 2(p_8)\,, \quad \text{and} \quad (p_1) + \cdots + (p_8)$$

are three effective divisors defining $(\mathscr{L}\otimes\mathscr{O}_{\mathbf{P}^2}(1)|_C)^{\otimes 2} = \mathscr{O}_{\mathbf{P}^2}(2)|_C$. Thus, one has

i) a contact conic q_1 such that $\operatorname{div}(q_1) = 2(p_1) + \cdots + 2(p_4)$,

ii) a contact conic q_2 such that $\operatorname{div}(q_2) = 2(p_5) + \cdots + 2(p_8)$, and

iii) a conic q such that $\operatorname{div}(q) = (p_1) + \cdots + (p_8)$.

Altogether, $q^2, q_1q_2 \in \Gamma(C, \mathscr{O}_{\mathbf{P}^2}(4)|_C)$ both have $2(p_1) + \cdots + 2(p_8)$ as its associated divisor. Therefore, they must agree up to a constant factor $c \in k^*$. In other words, $q^2 - cq_1q_2 = 0$ holds on C. One may normalise q_1 so that this equation takes the form

$$q^2 - q_1q_2 = 0\,. \tag{3}$$

In order to make sure that this is the equation of the curve C, one still needs to exclude that (3) holds identically on the whole of $\mathbf{P}^2$. Assuming it would hold identically, we first observe that q_1 and q_2 are not associates, since $\operatorname{div}(q_1) \neq \operatorname{div}(q_2)$. In view of this, equation (3) implies that q splits into two non-associate linear factors, $q = l_1l_2$. The equation $l_1^2l_2^2 = q_1q_2$ then shows, finally, that both q_1 and q_2 must define double lines, e.g. $q_1 \sim l_1^2$ and $q_2 \sim l_2^2$. But now $\operatorname{div}(q_1) = 2(p_1)+\cdots+2(p_4)$ yields $\operatorname{div}(l_1) = (p_1) + \cdots + (p_4)$, so $(p_1) + \cdots + (p_4)$ is a divisor defining $\mathscr{O}_{\mathbf{P}^2}(1)|_C$ and not $\mathscr{L}\otimes\mathscr{O}_{\mathbf{P}^2}(1)|_C$, a contradiction.

c) The conics $D_{(s:t)}\colon s^2q_1 + 2stq + t^2q_2 = 0$ are indeed contact conics of C. For $s = 0$, this was shown above. Otherwise, on $D_{(s:t)}$, one has $q_1 = -\frac{2t}{s}q - \frac{t^2}{s^2}q_2$, hence the equation of C takes the form $0 = q^2 - (-\frac{2t}{s}q - \frac{t^2}{s^2}q_2)q_2 = (q + \frac{t}{s}q_2)^2$. Moreover, the contact conics $D_{(s:t)}$ are all associated with the same 2-torsion invertible sheaf, as the base scheme $\mathbf{P}^1$ is connected. According to b), this sheaf is exactly $\mathscr{L}$.

On the other hand, we showed above that there is only a one-dimensional linear system of effective divisors that define $\mathscr{L}\otimes\mathscr{O}_{\mathbf{P}^2}(1)|_C$. Furthermore, since $2 < \deg C = 4$, the divisor determines the conic uniquely, which implies the claim.

d) A direct calculation shows that the conics

$$w = \tfrac{s}{t}q_1 + q, \quad s^2q_1 + 2stq + t^2q_2 = 0$$

lie on S, for $(s:t) \in \mathbf{P}^1$. For $t = 0$, this is supposed to mean $w = q$ and $q_1 = 0$. The second conic bundle is obtained replacing w by $-w$. $\square$

Remark 2.4. Part a) of this result is essentially [Do, Theorem 6.2.3], where it is deduced from the general framework of theta characteristics. For our purposes, this generality is neither necessary nor does it lead to additional clarity. On the other hand, we find the direct proof, just using the Riemann-Roch Theorem, quite instructive.

Corollary 2.5. *Let C be a nonsingular plane quartic over an algebraically closed field. Then the mapping $\Pi\colon \{$pairs of bitangents of $C\} \to \mathrm{Pic}(C)_2 \backslash \{\mathscr{O}_C\}$ is surjective and precisely six-to-one.*

Proof. Let $\mathscr{L} \in \mathrm{Pic}(C)_2 \backslash \{\mathscr{O}_C\}$ be any element. Then the contact conics associated with $\mathscr{L}$ form a one-dimensional family, which can be written in the form $D_{(s:t)}\colon s^2 q_1 + 2stq + t^2 q_2 = 0$. The quadratic forms q_1, q_2, and q occurring may be described by symmetric 3×3-matrices M_1, M_2, and M, respectively. Degenerate conics occur at the zeroes of $\det(s^2 M_1 + 2stM + t^2 M_2)$, which is a binary form that is homogeneous of degree six. Consequently, not more than six of the conics may be degenerate and not more than six pairs of bitangents may be mapped to $\mathscr{L}$ under Π. Since $378 = 63 \cdot 6$, the proof is complete. $\square$

Definition 2.6 (cf. [Do, Section 6.1.1])**.** Let $\mathscr{L} \in \mathrm{Pic}(C)_2 \backslash \{\mathscr{O}_C\}$ be any element. Then the preimage $\Pi^{-1}(\mathscr{L})$ is called a *Steiner hexad.*

2.7 (Degree two del Pezzo surfaces)**.** Let $C\colon Q = 0$ be a nonsingular plane quartic over a field k. Then, for every $\lambda \in k^*$, there is the double cover of $\mathbf{P}^2$, ramified at C, given by $S\colon \lambda w^2 = Q$. This is a del Pezzo surface of degree two [Ko96, Theorem III.3.5]. It is equipped with the projection $\pi\colon S \to \mathbf{P}^2$ and the Geiser involution

$$g\colon (w; T_0 : T_1 : T_2) \mapsto (-w; T_0 : T_1 : T_2)\,.$$

Over each of the 28 bitangents of $C_{\overline{k}}$, the double cover splits into two components. This yields exactly the 56 exceptional curves on $S_{\overline{k}}$. In fact, if $l = 0$ defines a bitangent then, modulo the linear form l, the equation of C is $cq^2 = 0$, for $c \in \overline{k}^*$ and a quadratic form q. Thus, the equation of S takes the form $\lambda w^2 = cq^2$, which shows the splitting into

$$w = \pm\sqrt{\frac{c}{\lambda}}q\,. \tag{4}$$

Remark 2.8 (The blown-up model). A del Pezzo surface of degree two over an algebraically closed field is isomorphic to $\mathbf{P}^2$, blown up in seven points $x_1, \dots, x_7$ in general position [Ma, Theorem 24.4.(iii)]. In the blown-up model, the 56 exceptional curves are given as follows, cf. [Ma, Theorem 26.2].

i) e_i, for $i = 1, \dots, 7$, the inverse image of the blow-up point x_i.

ii) l_{ij}, for $1 \leqslant i < j \leqslant 7$, the strict transform of the line through x_i and x_j. The class of l_{ij} in $\mathrm{Pic}(S)$ is $l - e_i - e_j$.

iii) $\widetilde{l}_{ij}$, for $1 \leqslant i < j \leqslant 7$, the strict transform of the conic through all blow-up points excluding x_i and x_j. The class of $\widetilde{l}_{ij}$ in $\mathrm{Pic}(S)$ is $2l - e_1 - \dots - e_7 + e_i + e_j$.

iv) $\widetilde{e}_i$, for $i = 1, \ldots, 7$, the strict transform of the unique singular cubic curve through all seven blow-up points that has x_i as a double point. The class of $\widetilde{e}_i$ in $\mathrm{Pic}(S)$ is $3l - e_1 - \cdots - e_7 - e_i$.

The Geiser involution g interchanges e_i with $\widetilde{e}_i$, for $i = 1, \ldots, 7$, and l_{ij} with $\widetilde{l}_{ij}$, for $1 \leqslant i < j \leqslant 7$. Furthermore, the maximal subgroup $\widetilde{G} \subset S_{56}$ that respects the intersection pairing operates transitively on systems of i mutually skew exceptional curves, for $i = 1, \ldots, 5$, or 7 [Ma, Corollary 26.8.(i)]. Thus, given such a system of i exceptional curves, one may assume without restriction that the curves are $e_1, \ldots, e_i$.

2.9 (The Picard group)**.** The Picard group $\mathrm{Pic}(S)$ of a degree two del Pezzo surface S over an algebraically closed field is a free abelian group of rank 8 and generated by the 56 exceptional curves. One has the transfer map $\pi_*\colon \mathrm{Pic}(S) \to \mathrm{Pic}(\mathbf{P}^2)$. The kernel $\mathrm{Pic}(S)[\pi_*]$ is

i) generated by all differences $\mathscr{O}_S(E - E')$ of exceptional curves,

ii) generated up to index three by $\mathscr{O}_S(E_2 - E_1), \ldots, \mathscr{O}_S(E_7 - E_1)$, and $\mathscr{O}_S(\widetilde{E}_1 - E_1)$, for seven mutually skew exceptional curves $E_1, \ldots, E_7$ and $\widetilde{E}_1 := g(E_1)$.

The second assertion easily follows from a short consideration in the blown-up model.

2.10 (The various restriction homomorphisms)**.** For the double cover S of $\mathbf{P}^2$, ramified at the nonsingular quartic C, one also has the restriction homomorphism $r\colon \mathrm{Pic}(S) \to \mathrm{Pic}(C)$. Formula (4) shows that, if E is an exceptional curve on S and $\pi(E)$ touches C in p_1 and p_2 then

$$r(\mathscr{O}_S(E)) = \mathscr{O}_C(\tfrac{1}{2}C.\pi(E)) = \mathscr{O}_C((p_1) + (p_2))\,.$$

Consequently, $r(\mathscr{O}_S(2E)) = \mathscr{O}_{\mathbf{P}^2}(1)|_C$, for every exceptional curve E.

Let us denote the restriction of r to $\mathrm{Pic}(S)[\pi_*]$ by $r'\colon \mathrm{Pic}(S)[\pi_*] \to \mathrm{Pic}(C)$. Then

$$r'(\mathscr{O}_S(E - E')) = \Pi(\{\pi(E), \pi(E')\})\,. \tag{5}$$

This shows that r' maps $\mathrm{Pic}(S)[\pi_*]$ only to $\mathrm{Pic}(C)_2$. Moreover, the homomorphism $r'\colon \mathrm{Pic}(S)[\pi_*] \to \mathrm{Pic}(C)_2$ is obviously onto. We have the induced surjective homomorphism

$$\overline{r}\colon\ \mathrm{Pic}(S)[\pi_*]/2\,\mathrm{Pic}(S)[\pi_*] \longrightarrow \mathrm{Pic}(C)_2\,. \tag{6}$$

Finally, recall that, via the Chern class homomorphism, there is a canonical isomorphism $\mathrm{Pic}(S)/2\,\mathrm{Pic}(S) \cong H^2_{\text{ét}}(S, \mathbb{Z}/2\mathbb{Z})$.

Lemma 2.11 (Criterion for Steiner hexads)**.** *Let $C \subset \mathbf{P}^2$ be a nonsingular plane quartic over an algebraically closed field and S be the double cover of $\mathbf{P}^2$, ramified at C. Furthermore, let $\{E_1, E_1', \ldots, E_6, E_6'\}$ be a set of twelve exceptional curves on S such that*

i) *$E_1, \ldots, E_6$ are mutually skew,*

ii) *$E_1', \ldots, E_6'$ are mutually skew, and*

iii) *$E_i \cdot E_j' = \delta_{ij}$, for $1 \leqslant i, j \leqslant 6$ and δ_{ij} the Kronecker symbol.*

Then $\{(\pi(E_1),\pi(E_1')),\dots,(\pi(E_6),\pi(E_6'))\}$ *is a Steiner hexad.*

Proof. The exceptional curves $E_1, E_1', \dots, E_6, E_6'$ generate, together with the canonical class $[K]$, the whole Picard group $\mathrm{Pic}(S)$ up to finite index. Indeed, the 13×13 intersection matrix

$$\begin{pmatrix} -1 & 1 & 0 & 0 & \dots & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & \dots & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & \dots & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & \dots & 0 & 0 & -1 \\ & & & & & & & \\ 0 & 0 & 0 & 0 & \dots & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & \dots & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 & \dots & -1 & -1 & 2 \end{pmatrix} \tag{7}$$

is of rank 8. (The index of $\langle E_1, E_1', \dots, E_6, E_6', [K]\rangle \subset \mathrm{Pic}(S)$ is, in fact, 2.)

Thus, we may establish the equality $\mathscr{O}_S(E_1 + E_1') = \dots = \mathscr{O}_S(E_6 + E_6')$ in $\mathrm{Pic}(S)$ by just noting that each of the six divisors has intersection number 0 with $E_1, E_1', \dots, E_6, E_6'$ and (-2) with $[K]$. Consequently,

$$r(\mathscr{O}_S(E_1 + E_1')) = \dots = r(\mathscr{O}_S(E_6 + E_6'))\,.$$

As $r(\mathscr{O}_S(2E)) = \mathscr{O}_{\mathbf{P}^2}(1)|_C$ for every exceptional curve, formula (5) yields that

$$\Pi(\{\pi(E_1),\pi(E_1')\}) = \dots = \Pi(\{\pi(E_6),\pi(E_6')\})\,.$$

Moreover, the bitangents $\pi(E_1),\pi(E_1'),\dots,\pi(E_6),\pi(E_6')$ are distinct, as the intersection number 2 is excluded, according to our assumptions. Thus, we indeed have a Steiner hexad. □

Remarks 2.12. i) Assume that S has a conic bundle with six split fibres $F_i = E_i \cup E_i'$, for $i = 1,\dots,6$. Then the assumptions of the criterion above are satisfied and hence $\{(\pi(E_1),\pi(E_1')),\dots,(\pi(E_6),\pi(E_6'))\}$ is a Steiner hexad.

ii) In particular, we may revert the conclusion of Proposition 2.3.d). Indeed, suppose that a double cover of $\mathbf{P}^2_k$, ramified at the nonsingular quartic C, has a k-rational conic bundle, with geometrically six degenerate fibres each splitting into two lines. Then every irreducible component E is automatically an exceptional curve, as one has $0 = EF = E(E + E') = E^2 + 1$, for F the class of the fibre. Therefore, the criterion applies and yields a Galois invariant Steiner hexad for C. As a consequence, there is a k-rational class in $\mathrm{Pic}(C)_2$, as well.

Corollary 2.13. *Let $C \subset \mathbf{P}^2$ be a nonsingular plane quartic over an algebraically closed field and L be a bitangent. Then the intersection of all Steiner hexads containing L consists only of L.*

Proof. We consider a double cover of $\mathbf{P}^2$, ramified at C, and work in the blown-up model. Without loss of generality, let us assume that $L = \pi(e_1)$. Then Lemma 2.11 yields the six Steiner hexads $\mathbf{H}_2,\dots,\mathbf{H}_7$ containing L, which are given by

$$\mathbf{H}_k := \{\pi(e_i), \pi(l_{ik}) \mid 1 \leqslant i \leqslant 7, i \neq k\}\,,$$

for $k = 2, \ldots, 7$. Note here that the assumption $i \neq k$ is necessary to have l_{ik} even defined. Furthermore, it ensures that $e_i \cdot l_{i'k} = 0$ for $i \neq i'$. Clearly, $L = \pi(e_1)$ is the only bitangent $\mathbf{H}_2, \ldots, \mathbf{H}_7$ all have in common. □

Proposition 2.14. *Let $C \subset \mathbf{P}^2$ be a nonsingular plane quartic over an algebraically closed field and S be the double cover of $\mathbf{P}^2$, ramified at C. We equip* $\mathrm{Pic}(S)[\pi_*]/2\,\mathrm{Pic}(S)[\pi_*]$ *with the $\mathbb{F}_2$-valued symplectic pairing, induced by the intersection pairing, and* $\mathrm{Pic}(C)_2$ *with the Weil pairing.*

a) *Then the restriction*

$$\overline{r}\colon \mathrm{Pic}(S)[\pi_*]/2\,\mathrm{Pic}(S)[\pi_*] \longrightarrow \mathrm{Pic}(C)_2$$

is a symplectic epimorphism.

b) *The kernel of $\overline{r}$ is the radical of* $\mathrm{Pic}(S)[\pi_*]/2\,\mathrm{Pic}(S)[\pi_*]$, *which is generated by the class of $\mathscr{O}_S(\widetilde{E} - E)$, for an arbitrary exceptional curve E and $\widetilde{E} := g(E)$.*

Proof. First of all, let $E_1, \ldots, E_7$ be mutually skew exceptional curves and put $\widetilde{E}_1 := g(E_1)$. As seen in 2.9.ii), up to the odd index three, $\mathrm{Pic}(S)[\pi_*]$ is generated by the sheaves $\mathscr{O}_S(E_2 - E_1), \ldots, \mathscr{O}_S(E_7 - E_1)$, and $\mathscr{O}_S(\widetilde{E}_1 - E_1)$. Their intersection matrix is, obviously

$$\begin{pmatrix} -2 & -1 & -1 & -1 & -1 & -1 & -2 \\ -1 & -2 & -1 & -1 & -1 & -1 & -2 \\ -1 & -1 & -2 & -1 & -1 & -1 & -2 \\ -1 & -1 & -1 & -2 & -1 & -1 & -2 \\ -1 & -1 & -1 & -1 & -2 & -1 & -2 \\ -1 & -1 & -1 & -1 & -1 & -2 & -2 \\ -2 & -2 & -2 & -2 & -2 & -2 & -6 \end{pmatrix}.$$

Moreover, the class of $\mathscr{O}_S(\widetilde{E}_1 - E_1)$ is clearly an element in the kernel of $\overline{r}$.

Thus, one only has to show that, for $v_i := r(\mathscr{O}_S(E_i - E_1))$, the Weil pairing $\langle v_i, v_j \rangle$ evaluates nontrivially whenever $i \neq j$, for $i, j \in \{2, \ldots, 7\}$. In order to do this, let us recall that the Riemann-Mumford relations [Do, Theorem 5.1.1] ensure that

$$\langle \varepsilon, \delta \rangle = h^0(C, \varepsilon + \delta + \eta) + h^0(C, \varepsilon + \eta) + h^0(C, \delta + \eta) + h^0(C, \eta) \pmod 2\,,$$

for $\eta \in \mathrm{Pic}(C)$ any invertible sheaf such that $\eta^{\otimes 2} \cong \mathscr{O}_{\mathbf{P}^2}(1)|_C$. Therefore, taking $\eta := r(\mathscr{O}_S(E_1))$, we find

$$\begin{aligned} \langle v_i, v_j \rangle = {} & h^0(C, r(\mathscr{O}_S(E_i + E_j - E_1))) \\ & + h^0(C, r(\mathscr{O}_S(E_i))) + h^0(C, r(\mathscr{O}_S(E_j))) + h^0(C, r(\mathscr{O}_S(E_1))) \pmod 2\,. \end{aligned}$$

But $h^0(C, r(\mathscr{O}_S(E_i))) = h^0(C, r(\mathscr{O}_S(E_j))) = h^0(C, r(\mathscr{O}_S(E_1))) = 1$, as shown in (2), so we only need to verify that

$$h^0(C, r(\mathscr{O}_S(E_i + E_j - E_1))) = 0\,.$$

This is exactly Lemma 2.16 below. □

Remarks 2.15. i) The fact that $\overline{r}$ is symplectic has been stated for the first time in [DO, Ch. IX, Sec. 1, Lemma 2]. Our proof follows the ideas of [Za, Remark 2.11] and is very different from the original one.

ii) In much more generality, A. N. Skorobogatov [Sk17, Corollary 3.2.iii)] constructs a canonical homomorphism Φ the other way round and shows that there is a short exact sequence

$$0 \longrightarrow H^1_{\text{ét}}(C, \mathbb{Z}/2\mathbb{Z}) \stackrel{\Phi}{\longrightarrow} H^2_{\text{ét}}(S, \mathbb{Z}/2\mathbb{Z})/\pi^* H^2_{\text{ét}}(\mathbf{P}^2, \mathbb{Z}/2\mathbb{Z}) \stackrel{\pi_*}{\longrightarrow} H^2_{\text{ét}}(\mathbf{P}^2, \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

of Galois modules. The homomorphism Φ is given by any section of the homomorphism $H^2_{\text{ét}}(S, \mathbb{Z}/2\mathbb{Z})[\pi_*] \to H^1_{\text{ét}}(C, \mathbb{Z}/2\mathbb{Z})$ that is induced by the restriction $\overline{r}$ [Sk17, Lemma 3.3]. It is, unfortunately, not discussed in [Sk17] whether or in which generality Φ is symplectic.

Lemma 2.16. *Let $C \subset \mathbf{P}^2$ be a nonsingular plane quartic over an algebraically closed field and S the double cover of $\mathbf{P}^2$, ramified at C. Then, for three mutually skew exceptional curves E, E', and E'' on S, one has*

$$h^0(C, r(\mathscr{O}_S(E + E' - E''))) = 0\,. \tag{8}$$

Proof. The only way for $r(\mathscr{O}_S(E + E' - E''))$ to have a section is that there is a fourth bitangent L of C such that $\Pi(\{L, \pi(E)\}) = \Pi(\{\pi(E'), \pi(E'')\})$. Indeed, in a notation analogous to that used before Remark 2.2, one has

$$\mathscr{L} := r(\mathscr{O}_S(E' - E'')) = \mathscr{O}_C((p'_1) + (p'_2) - (p''_1) - (p''_2)) \in \operatorname{Pic}(C)_2\,,$$

and a section of $r(\mathscr{O}_S(E + E' - E''))$ would yield two points p'''_1 and $p'''_2 \in C$ such that $\mathscr{O}_C((p'''_1) + (p'''_2) - (p_1) - (p_2)) = \mathscr{L}$, too, for p_1 and p_2 the points of tangency of $\pi(E)$ with C. In particular, the divisor

$$D := 2(p'''_1) + 2(p'''_2) - 2(p_1) - 2(p_2) \in \operatorname{Div}(C)$$

would be principal. However, $\Gamma(C, \mathscr{O}_C(2(p_1) + 2(p_2))) = \langle \frac{T_0}{s}, \frac{T_1}{s}, \frac{T_2}{s} \rangle$, for s the linear form defining the bitangent $\pi(E)$, according to formula (1). In particular, one must have $D = \operatorname{div}(l/s)$ for a further linear form l. But then, $L := Z(l)$ is clearly a bitangent touching C in p'''_1 and p'''_2.

Take an exceptional curve E''' on S such that $\pi(E''') = L$. As r' induces an epimorphism $\overline{r}\colon \operatorname{Pic}(S)[\pi_*]/2\operatorname{Pic}(S)[\pi_*] \longrightarrow \operatorname{Pic}(C)_2$ with kernel generated by the class of $\mathscr{O}_S(\widetilde{E} - E)$, we find that

$$\begin{aligned} \mathscr{O}_S(E''' - E) &\equiv \mathscr{O}_S(E' - E'') \qquad (\operatorname{mod} \langle 2\operatorname{Pic}(S)[\pi_*], \mathscr{O}_S(\widetilde{E} - E)\rangle)\,, \qquad \text{i.e.}\,, \\ \mathscr{O}_S(E''') &\equiv \mathscr{O}_S(E + E' + E'') \quad (\operatorname{mod} \langle 2\operatorname{Pic}(S)[\pi_*], \mathscr{O}_S(\widetilde{E} - E)\rangle)\,. \end{aligned}$$

In order to complete the argument, let us work in the blown-up model. One may assume without restriction that $E = e_1$, $E' = e_2$, and $E'' = e_3$. Then E''' must have odd intersection number (-1 or 1) with e_1, e_2, and e_3 and even intersection number (0 or 2) with $e_4, \ldots, e_7$, or the other way round. A short consideration shows, however, that such an exceptional curve does not exist. $\square$

Remark 2.17. Formula (8) is classically known, as $\pi(E) = \pi(e_1)$, $\pi(E') = \pi(e_2)$, and $\pi(E'') = \pi(e_3)$ form part of the so-called Aronhold set $\{\pi(e_1), \ldots, \pi(e_7)\}$, cf. [Do, Section 6.1.2].

Corollary 2.18 (The admissible subgroup in degree 28 versus that in degree 56). *Let C be a nonsingular plane quartic over an algebraically closed field and S be the double cover of $\mathbf{P}^2$, ramified at C. Write $\widetilde{\Omega}_C$ for the set of the 56 exceptional curves and $\widetilde{G} \subset \operatorname{Sym}\widetilde{\Omega}_C \cong S_{56}$ for the maximal subgroup that respects the intersection pairing on S. Moreover, put*

$$G := \{\sigma \in \operatorname{Sym}\Omega_C \mid \textit{The permutation } \sigma \textit{ of the 28 bitangents respects the Steiner hexads, and the operation on the 63 Steiner hexads induces a symplectic automorphism of } H^1_{\text{ét}}(C, \mathbb{Z}/2\mathbb{Z})\}\,.$$

Then

a) $\widetilde{G} \cong W(E_7)$ *and* $G \cong \operatorname{Sp}_6(\mathbb{F}_2)$.

b) *The group $\widetilde{G}$ has the pairs of exceptional curves, lying over the same bitangent, as a block system. The permutation representation $\iota\colon W(E_7)/Z \to \operatorname{Sym}\Omega_C \cong S_{28}$ induced on the blocks is faithful.*

c) *The image of ι is exactly G.*

Proof. First, the canonical homomorphism $G \to \operatorname{Aut}(H^1_{\text{ét}}(C, \mathbb{Z}/2\mathbb{Z})) \cong \operatorname{Sp}_6(\mathbb{F}_2)$ is injective. Indeed, for a permutation $\sigma \in G \subset \operatorname{Sym}\Omega_C \cong S_{28}$ to lie in the kernel, the induced operation $\sigma^{(2)} \in \operatorname{Sym}\Omega_C^{(2)} \cong S_{\frac{28\cdot 27}{2}}$ on 2-sets must keep all Steiner hexads in place. According to Corollary 2.13, this yields $\sigma^{(2)} = \mathrm{id}$, which suffices for $\sigma = \mathrm{id}$.

On the other hand, the fact that $\widetilde{G} \cong W(E_7)$ is shown in [Ma, Theorem 23.9]. The operation of $\widetilde{G}$ respects the pairs $\{E, \widetilde{E}\}$ as these are the only ones with intersection number 2. Moreover, the centre, which is generated by the Geiser involution, clearly fixes all blocks, such that there is an induced permutation representation $\iota\colon W(E_7)/Z \to \operatorname{Sym}\Omega_C$.

A nontrivial element in kernel of ι would be represented by some $\tau \in W(E_7)$ that flips some but not all of the pairs $\{E, \widetilde{E}\}$. The projection formula shows, however, that, for arbitrary exceptional curves E and E', one has

$$EE' + E\widetilde{E}' = E(E' + \widetilde{E}') = E \cdot \pi^*L = \pi_*E \cdot L = 1\,, \tag{9}$$

for $\widetilde{E}' = g(E')$. The assumption that τ would act in the way that $E \mapsto E$, $E' \mapsto \widetilde{E}'$, and $\widetilde{E}' \mapsto E'$ is hence contradictory, as one has, without restriction, $EE' = 0$ and $E\widetilde{E}' = 1$. Thus, we see that ι is injective, which completes the proof of b).

Finally, the operation of the group $W(E_7)$ on $\operatorname{Pic}(S)$, and hence on $\operatorname{Pic}(S)[\pi_*]$, respects the intersection pairing. Consequently, the $\mathbb{F}_2$-valued symplectic pairing on $\operatorname{Pic}(S)[\pi_*]/2\operatorname{Pic}(S)[\pi_*]$ is respected, too. Therefore, Proposition 2.14.a) yields that $W(E_7)$ operates on $\operatorname{Pic}(C)_2 \cong H^1_{\text{ét}}(C, \mathbb{Z}/2\mathbb{Z})$ via symplectic automorphisms. In particular, the homomorphism $\iota\colon W(E_7)/Z \to \operatorname{Sym}\Omega_C$ factors via G, and the following diagram commutes,

$$\begin{array}{ccccc}
\operatorname{Sym}\Omega_C \cong S_{28} & = & \operatorname{Sym}\Omega_C \cong S_{28} & & \\
\uparrow & & \uparrow & & \\
W(E_7)/Z & \xrightarrow{\overline{\iota}} & G & \hookrightarrow & \operatorname{Sp}_6(\mathbb{F}_2)\,.
\end{array}$$

Altogether, we have two injections $W(E_7)/Z \hookrightarrow G \hookrightarrow \operatorname{Sp}_6(\mathbb{F}_2)$ in a row. As both groups, $W(E_7)/Z$ and $\operatorname{Sp}_6(\mathbb{F}_2)$, are of order 1 451 520, the injections must be bijective. In other words, c) is shown and the proof of a) is complete. ☐

Corollary 2.19. *Define the subgroup $U_{63} \subset G$ as the stabiliser of a Steiner hexad. Then*

a) *The subgroup U_{63} is uniquely determined up to conjugation in G and of index* 63*. As a permutation group in degree 28, U_{63} is intransitive of orbit type* $[12, 16]$.

b) *As an abstract group, U_{63} is isomorphic to $(S_2 \wr S_6) \cap A_{12}$, operating on the size twelve orbit in the obvious way. In particular, the operation of an element $\sigma \in U_{63}$ on the size twelve orbit determines σ completely.*

Proof. Let us consider a non-singular model quartic $C \subset \mathbf{P}^2$ over an algebraically closed field and put S to be the double cover of $\mathbf{P}^2$, ramified at C.

a) According to Definition 2.6, Steiner hexads are in bijection with the 63 nonzero elements in $\operatorname{Pic}(C)_2$. Moreover, the group $G \cong \operatorname{Sp}_6(\mathbb{F}_2)$ operates transitively on those nonzero elements and therefore as well on the 63 Steiner hexads. This implies the two first assertions.

The final one is easily checked by an experiment in `magma` [BCP], which shows the following. The, up to conjugation, unique index-63 subgroup of the only simple permutation group of order 1 451 520 in degree 28 has orbit type $[12, 16]$.

b) Once again, the subgroup $U_{63} \subset G$ distinguishes a Steiner hexad and hence a nonzero element of $\operatorname{Pic}(C)_2$. According to Proposition 2.3.d), S has an associated conic bundle. As before, let us write $E_1, E_1', \ldots, E_6, E_6'$ for the irreducible components of its six split fibres. Then the distinguished Steiner hexad is simply $\{(\pi(E_1), \pi(E_1')), \ldots, (\pi(E_6), \pi(E_6'))\}$. Having numbered these twelve bitangents from 1 to 12, the operation of U_{63} defines a homomorphism $i\colon U_{63} \to S_{12}$.

Injectivity. Assume that $\sigma \in U_{63}$ fixes each of the twelve bitangents. Then a lift $\widetilde{\sigma} \in \widetilde{G}$ of σ either fixes $E_1, E_1', \ldots, E_6, E_6'$ or sends each of these exceptional curves to its image under the Geiser involution. Without restriction, we may assume that $\widetilde{\sigma}$ fixes the curves.

Next, we observe that the sum over all 56 exceptional curves in $\operatorname{Pic}(S)$ is $-28[K]$. This shows that any permutation in $\operatorname{Sym}\widetilde{\Omega}_C \cong S_{56}$ fixes the canonical class $[K]$. Moreover, as seen in (7), together with $[K]$, the curves $E_1, E_1', \ldots, E_6, E_6'$ generate $\operatorname{Pic}(S)$ up to a finite index. Consequently, $\widetilde{\sigma}$ operates identically on the whole of $\operatorname{Pic}(S)$. It must therefore fix each of the 56 exceptional curves, which shows that $\widetilde{\sigma} = \mathrm{id}$.

Image. The maximal permutation group that respects the intersection matrix of $E_1, E'_1, \ldots, E_6, E'_6$ is clearly the wreath product $S_2 \wr S_6$. However, the Faddeev reciprocity law for conic bundles [Sk13, Corollary 3.3] implies that Galois operations permuting these twelve curves are limited to even permutations. We claim that, for a model surface, the action of U_{63} is provided by a Galois operation. Indeed, the results of T. Shioda [Shi, Section 3] and R. Erné [Er, Corollary 3] provide a nonsingular plane quartic over $\mathbb{Q}$ of the kind that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ operates on the bitangents via G. Hence, for a certain number field l_{63} of degree 63, $\mathrm{Gal}(\overline{\mathbb{Q}}/l_{63})$ acts precisely via $U_{63} \subset G$. Therefore, the image of $i\colon U_{63} \to S_{12}$ is contained in $(S_2 \wr S_6) \cap A_{12}$. As both groups, U_{63} and $(S_2 \wr S_6) \cap A_{12}$, are of order 23 040, equality holds. □

3. Conic bundles with prescribed Galois operation I– The construction

Let a field k, a Galois extension l/k, and an injection $g := \mathrm{Gal}(l/k) \stackrel{i}{\hookrightarrow} S_2 \wr S_6$ be given. Then one has the subgroups

$$B := \mathrm{im}(\mathrm{pr}\circ i\colon g \to S_6) \subseteq S_6 \quad \text{and} \quad H := \ker(\mathrm{pr}\circ i\colon g \to S_6) \subseteq g\,.$$

The normal subgroup $H \subseteq g$ defines an intermediate field k' such that $\mathrm{Gal}(k'/k) \cong B$ and $\mathrm{Gal}(l/k') = H$. As $H \subseteq (\mathbb{Z}/2\mathbb{Z})^6$, the latter is a Kummer extension. I.e., one has $l = k'(\sqrt{A})$, for a subgroup $A \subseteq k'^*/(k'^*)^2$. The lemmata below analyse this situation from the field-theoretic point of view.

For this, let us note that the group $g \stackrel{i}{\hookrightarrow} S_2 \wr S_6$ naturally operates on twelve objects $1_a, 1_b, \ldots, 6_a, 6_b$ forming six pairs. For every $n \in \{1, \ldots, 6\}$, the stabiliser $\mathrm{Stab}_g(n_a) = \mathrm{Stab}_g(n_b)$ is a subgroup of index 1 or 2 in $\mathrm{Stab}_g(\{n_a, n_b\})$, hence normal.

Let us denote the subfield of l, corresponding to $\mathrm{Stab}_g(\{n_a, n_b\})$ under the Galois correspondence, by k_n. Similarly, we write l_n for the subfield corresponding to $\mathrm{Stab}_g(n_a)$. According to these definitions, one has $k_n \subseteq k'$ and $k_n \subseteq l_n$, the latter extension being of degree 1 or 2. Moreover, $k' = k_1 \cdots k_6$ and $l = l_1 \cdots l_6$.

Sublemma 3.1. *Let k be a field, $d > 0$ be an integer, and l an étale k-algebra of degree d.*

a) *Then, for every $x \in l^*$, there is a dominant morphism $q_x\colon \mathbf{A}(l) \to \mathbf{A}(l)$ of k-schemes that induces on k-rational points the mapping $l \to l$, $t \mapsto xt^2$.*

b) *There is a dominant morphism $c_l\colon \mathbf{A}(l) \to \mathbf{A}(k[T]_d)$ of k-schemes inducing on k-rational points the mapping $l \to k[T]_d$, $t \mapsto \chi_t$, for χ_t the characteristic polynomial of the multiplication map $\cdot t\colon l \to l$.*

c) *Let d_1 and d_2 be positive integers such that $d_1 + d_2 = d$. Then there is a dominant morphism $m_{d_1,d_2}\colon \mathbf{A}(k[T]_{d_1}) \times \mathbf{A}(k[T]_{d_2}) \to \mathbf{A}(k[T]_d)$ of k-schemes that induces on k-rational points the mapping $k[T]_{d_1} \times k[T]_{d_2} \to k[T]_d$, $(f_1, f_2) \mapsto f_1 f_2$.*

Proof. The mappings given are clearly polynomial, so they define morphisms of k-schemes.

Moreover, dominance may be tested after base extension to the algebraic closure. Then, $l \cong k^d$. Thus, by inspecting k-rational points, one finds that all three types of morphisms are always quasi-finite, with generic fibres of sizes 2^d, $d!$, and $\binom{d}{d_1}$, respectively. As source and target are of the same dimension, this suffices for dominance. $\square$

Lemma 3.2. *Let a Galois extension l/k and an injection $\mathrm{Gal}(l/k) \hookrightarrow S_2 \wr S_6$ be given, and let A, B, and k' have the same meaning as above.*

a) *Then there is a natural ordered set $(\alpha_1, \ldots, \alpha_6)$ contained in $A \subseteq k'^*/(k'^*)^2$ such that*

i) *The elements $\alpha_1, \ldots, \alpha_6$ generate the abelian group A.*

ii) *The Galois group $\mathrm{Gal}(k'/k) \cong B$ permutes $\alpha_1, \ldots, \alpha_6 \in k'^*/(k'^*)^2$ exactly as described by the natural inclusion $B \subseteq S_6$.*

b) *There exist lifts $A_1, \ldots, A_6 \in k'^*$ of $\alpha_1, \ldots, \alpha_6$ that are still permuted by $\mathrm{Gal}(k'/k)$.*

Proof. Let us note that $k'^*/(k'^*)^2$ is naturally acted upon by $\mathrm{Gal}(k'/k)$. Indeed, an automorphism of k' sends $(k'^*)^2$ to itself.

a) The group $\mathrm{Gal}(l/k') = \mathrm{Gal}(k'(\sqrt{A})/k') = H$ is abelian of exponent 2. Hence, according to Kummer theory, H is the dual of the group $A \subseteq k'^*/(k'^*)^2$ and, consequently, $A \cong H^\vee$ holds as well. Furthermore, the standard linear forms on $H \subseteq (\mathbb{Z}/2\mathbb{Z})^6$,

$$e_i : (z_1, \ldots, z_6) \mapsto z_i \, ,$$

for $i = 1, \ldots, 6$, generate $H^\vee$ and are permuted by $\mathrm{Gal}(k'/k) \cong B \subseteq S_6$ in the natural manner. Thus, the elements $\alpha_1, \ldots, \alpha_6 \in A \cong H^\vee$, corresponding to these linear forms, satisfy conditions i) and ii).

b) The assertion simply means that each α_n may be lifted to k'^* in such a way that the size of its g-orbit remains the same. I.e., such that $\mathrm{Stab}_g(A_n) = \mathrm{Stab}_g(\alpha_n)$. Indeed, if $B \subseteq S_6$ is transitive then one may simply lift α_1 in this way, and take the $B = \mathrm{Gal}(k'/k)$-orbit of the lift. In the intransitive case, the same has to be done separately for each orbit in $\{1, \ldots, 6\}$.

In order to verify liftability of this particular kind for one of the α_n, recall that the classes $\alpha_1, \ldots, \alpha_6 \in k'^*/(k'^*)^2$ are characterised by the property that $\pm\sqrt{\alpha_1}$, $\ldots$, $\pm\sqrt{\alpha_6}$ are acted upon by g in the same way as the symbols $1_a, 1_b, \ldots, 6_a, 6_b$. Thus, we need elements $\sqrt{A_n} \in l$ of the kind that

$$\mathrm{Stab}_g(\sqrt{A_n}) = \mathrm{Stab}_g(n_a) \, . \tag{10}$$

Then clearly $\mathrm{Stab}_g(A_n) = \mathrm{Stab}_g(\{n_a, n_b\}) = \mathrm{Stab}_g(\alpha_n)$.

For this, we know that l_n is an at most quadratic extension of k_n. Hence, $l_n = k_n(\sqrt{A_n})$, for some $A_n \in k_n^*$. As the fields k_n are permuted by $\mathrm{Gal}(k'/k)$ in the same way as the sets $\{n_a, n_b\}$, at least as long as it is a primitive element of the field k_n, A_n has the property (10) required. $\square$

Lemma 3.3. *Let a Galois extension l/k and an injection $\mathrm{Gal}(l/k) \hookrightarrow S_2 \wr S_6$ be given, and let α_n, A_n, and k' be as above. Choose lifts $A_1, \ldots, A_6 \in k'^*$ of $\alpha_1, \ldots, \alpha_6 \in k'^*/(k'^*)^2$ that form a $\mathrm{Gal}(k'/k)$-invariant set.*

a) *Then the polynomial* $F \in k[T]$ *such that*

$$F(T) := (T - A_1) \cdots (T - A_6)$$

has splitting field k'.

b) *Furthermore,* $k(\sqrt{A_1}, ..., \sqrt{A_6}) = l$. *I.e.,* l *is the splitting field of* $F(T^2)$. *Finally, the operation of* $\mathrm{Gal}(l/k)$ *on the roots* $\pm\sqrt{A_i}$ *agrees with the natural operation of* $S_2 \wr S_6$ *on twelve objects forming six pairs.*

Proof. a) By construction, the polynomial $F \in k'[T]$ is $\mathrm{Gal}(k'/k)$-invariant, hence $F \in k[T]$. On the other hand, the splitting field of F is $k(A_1, \ldots, A_6)$, which is clearly contained in k'. The claim follows, since the Galois group of F is $B \subseteq A_6$ and hence coincides with that of k'.

b) The splitting field of $F(T^2)$ is $k(\sqrt{A_1}, ..., \sqrt{A_6})$, which is equal to l, according to our construction. The final assertion is obvious. □

Corollary 3.4. *Let a Galois extension* l/k *and an injection* $\mathrm{Gal}(l/k) \hookrightarrow S_2 \wr S_6$ *be given, where the field* k *is infinite. Then the set of all polynomials* F *as in Lemma 3.3 is Zariski dense in* $\mathbf{A}(k[T]_6)$.

Proof. Let $O_1, \ldots, O_m \subseteq \{1, \ldots, 6\}$ be the orbits under the operation of $B \subseteq S_6$. For each of them, let us choose a representative $n_i \in O_i$ and a lift $A_{n_i}^{(0)} \in k_{n_i}^*$. Then the polynomials in the sense above certainly include those of type

$$\prod_{i=1}^{m} \chi_{A_{n_i}^{(0)} \cdot t_i^2}(T) \,,$$

for all $t_i \in k_{n_i}^*$ such that $A_{n_i}^{(0)} \cdot t_i^2 \in k_{n_i}$ is a primitive element.

In order to prove Zariski density, let, at first, $i \in \{1, \ldots, m\}$ be arbitrary. Then, as $k_{n_i} \supseteq k$ is an infinite field, $k_{n_i} = \mathbf{A}(k_{n_i})(k)$ is Zariski dense in $\mathbf{A}(k_{n_i})$. Thus, Sublemma 3.1.a) shows that the elements

$$q_{A_{n_i}^{(0)}}(t_i) = A_{n_i}^{(0)} \cdot t_i^2 \in k_{n_i} = \mathbf{A}(k_{n_i})(k) \,,$$

for $t_i \in k_{n_i}^*$, form a Zariski dense subset in $\mathbf{A}(k_{n_i})$, too. The same is still true for the subset of these elements that are primitive. Indeed, being a primitive element is a Zariski open condition. Consequently, by Sublemma 3.1.b), the polynomials

$$c_{k_{n_i}}(A_{n_i}^{(0)} \cdot t_i^2) = \chi_{A_{n_i}^{(0)} \cdot t_i^2}(T) \in k[T]_{\#O_{n_i}} = \mathbf{A}(k[T]_{\#O_{n_i}})(k)$$

are Zariski dense in $\mathbf{A}(k[T]_{\#O_{n_i}})$. Part c) of Sublemma 3.1 finally implies the claim. □

Proposition 3.5 (A conic bundle with six split fibres and prescribed Galois action). *Let* k *be a field and* $F(T) = T^6 + a_5T^5 + \cdots + a_1T + a_0 \in k[T]$ *be a separable, monic polynomial of degree six, where* $a_0 = c^2$, *for some* $c \in k^*$.

a) *Then there exist exactly two pairs* (g,h) *of binary quadratic forms such that*

$$\det M(1,T) = -F(T)\,,$$

for

$$M(s,t) := \begin{pmatrix} -st+t^2 & st & g(s,t) \\ st & s^2 & t^2 \\ g(s,t) & t^2 & h(s,t) \end{pmatrix}. \tag{11}$$

For these, one has $g(1,0) = \pm c$.

b) *Let* (g,h) *be one of the pairs as in a), that with* $g(1,0) = c$. *Then the equation*

$$(T_0\, T_1\, T_2)\, M(s,t) \begin{pmatrix} T_0 \\ T_1 \\ T_2 \end{pmatrix} = 0 \tag{12}$$

defines a hyperplane $B_{F,c}$ *of bidegree* $(2,2)$ *in* $\mathbf{P}^1 \times \mathbf{P}^2$.

i) *The generic fibre of the projection* $\mathrm{pr}_1\colon B_{F,c} \to \mathbf{P}^1$ *is a conic. Degenerate fibres occur exactly over the roots of* F.

ii) *Each of the six reducible fibres geometrically splits into two lines. The irreducible components are defined over the splitting field* l *of* $F(T^2)$ *and permuted by* $\mathrm{Gal}(l/k)$ *in the same way as the roots of* $F(T^2)$.

iii) *The projection* $\mathrm{pr}_2\colon B_{F,c} \to \mathbf{P}^2$ *is a double cover of* $\mathbf{P}^2$, *ramified at a quartic curve* $C_{F,c}$. *For a generic choice of the polynomial* F, *the curve* $C_{F,c}$ *is nonsingular.*

Proof. a) A direct calculation shows that the determinant of $M(1,T)$ is exactly

$$-Th(1,T) + 2T^3 g(1,T) - g^2(1,T) - (T^2-T)T^4 = -Th(1,T) + T^5 - [T^3 - g(1,T)]^2\,. \tag{13}$$

Writing $g(1,T) = g_2T^2 + g_1T + g_0$, this term takes the form

$$-Th(1,T) - T^6 + (2g_2+1)T^5 - (g_2^2 - 2g_1)T^4 + \cdots - g_0^2\,,$$

so F determines the coefficients $g_2 := -\frac{a_5+1}{2}$ and $g_1 := \frac{-a_4+g_2^2}{2}$ uniquely and the absolute term $g_0 := \pm\sqrt{a_0} = \pm\sqrt{c^2}$ up to sign. In particular, $g(1,0) = g_0 = \pm c$. Finally, for either choice of g, the coefficients of (13) at T^i, for $i = 1$, 2, and 3, uniquely determine the quadratic polynomial $h(1,T)$ to be

$$h(1,T) := \frac{F - [T^3 - g(1,T)]^2 + T^5}{T}\,.$$

b.i) Split fibres correspond to zeroes of the determinant of $M(s,t)$.

ii) The first claim is that $\operatorname{rk} M(s,t) = 2$ for each $(s:t) \in \mathbf{P}^1$ such that $\det M(s,t) = 0$. For this, we observe that the upper left 2×2-minor of $M(s,t)$ is $(-s^3t)$, which vanishes only at 0 and ∞. However, all split fibres occur over points on the affine line and 0 is excluded, since $F(0) = a_0 = c^2 \neq 0$.

Finally, let t_0 be any root of F. Then the fibre $(B_{F,c})_{(1:t_0)}$ is the degenerate conic defined over $k(t_0)$ that is given by the 3×3-matrix $M(1,t_0)$. As $M(1,t_0)$ has vanishing determinant and a principal 2×2-minor of $(-t_0)$, the fibre $(B_{F,c})_{(1:t_0)}$ is projectively equivalent to the degenerate conic, given by $T_0^2 - t_0 T_1^2 = 0$. This latter conic clearly splits into two lines over $k(\sqrt{t_0})$.

For $t_1, \ldots, t_6$ the roots of F, the field of definition of the twelve irreducible components is hence $k(\sqrt{t_1}, \ldots, \sqrt{t_6})$, which is exactly the splitting field of $F(T^2)$. The final claim is obvious.

iii) The left hand side of equation (12) may be considered as a binary quadratic form, the coefficients of which are quadratic forms in T_0, T_1, and T_2, i.e. as

$$Q_0(T_0, T_1, T_2)s^2 + Q_1(T_0, T_1, T_2)st + Q_2(T_0, T_1, T_2)t^2 .$$

As such, its discriminant is the ternary quartic form $\boldsymbol{Q}_{F,c} := Q_1^2 - 4Q_0Q_2$. Thus, $C_{F,c}$ is the vanishing locus of $\boldsymbol{Q}_{F,c}$.

Using `magma` [BCP], it is easy to write down $\boldsymbol{Q}_{F,c}$ explicitly, in terms of the parameters $a_1, \ldots, a_5$, and c. We used the first author's code [El] for calculating the Dixmier–Ohno invariants of ternary quartic forms and found that the discriminant of $\boldsymbol{Q}_{F,c}$ is not the zero polynomial. □

Remarks 3.6. i) The quartic form $\boldsymbol{Q}_{F,c}$ occurs to us in the symmetric determinantal form $Q_1^2 - 4Q_0Q_2$. This is related to the fact that $C_{F,c}$ has a distinguished Steiner hexad, respectively a distinguished element in $\mathrm{Pic}(C_{F,c})_2 \backslash \{\mathscr{O}_{C_{F,c}}\}$, cf. Proposition 2.3.a). The quadratic forms Q_0 and Q_2 define particular contact conics of $C_{F,c}$.

ii) One may adopt the relative point of view, according to which formula (12) actually describes a family $\kappa\colon \boldsymbol{C} \to \operatorname{Spec} \mathbb{Q}[c, a_1, \ldots, a_5] = \mathbf{A}^6_{\mathbb{Q}}$ of plane quartics.

Then the discriminant of $\boldsymbol{Q}_{F,c}$ is a polynomial in $\mathbb{Q}[c, a_1, \ldots, a_5]$ that splits into two factors. One of them is exactly the discriminant of the degree six polynomial F. This coincidence is, of course, not at all surprising, since multiple zeroes of F cause a degenerate conic bundle.

The other factor, entering the discriminant of $\boldsymbol{Q}_{F,c}$ quadratically, reflects the fact that the total scheme $\boldsymbol{C}$ of the family κ is singular. The image $\Delta_{\mathrm{sing}} := \kappa(\boldsymbol{C}_{\mathrm{sing}})$ of the singular locus under projection to the parameter scheme $\mathbf{A}^6$ is the divisor defined by this factor. The corresponding fibres generically have only one singular point that is an ordinary double point.

Remark 3.7. The discriminant of the polynomial $F(T^2)$ is, up to square factors, $(-1)^{\deg F} F(0)$. In particular, in our situation, the discriminant is a square in k, such that one automatically has an injection

$$i\colon \mathrm{Gal}(l/k) \hookrightarrow (S_2 \wr S_6) \cap A_{12} .$$

Indeed, the general formula for the discriminants in a tower of fields (cf. [Ne, Corollary III.2.10]) shows

$$\Delta_{F(T^2)} = \Delta_F^2 \cdot N_{(k[T]/(F))/k}(r) ,$$

for r a root of F. Therefore, $\Delta_{F(T^2)} = \Delta_F^2 \cdot (-1)^{\deg F} F(0)$. But, in our case, one has $(-1)^{\deg F} F(0) = F(0) = a_0 = c^2$, which shows the claim.

4. Conic bundles with prescribed Galois operation II– The monodromy

4.1. Let an injection $i\colon \operatorname{Gal}(l/k) \hookrightarrow (S_2 \wr S_6) \cap A_{12}$ be given and F be a polynomial of the kind that l is the splitting field of $F(T^2)$. Then Proposition 3.5.b.ii) yields a numbering of the split fibres of the conic bundle surface $B_{F,c}$ from 1 to 6, such that $\operatorname{Gal}(l/k)$ operates on the twelve irreducible components as described by i. This shows that, when a numbering is given on the split fibres, then the operation of $\operatorname{Gal}(l/k)$ on the twelve irreducible components agrees with that described by i only up to an inner automorphism of $S_2 \wr S_6$.

According to Lemma 2.11 and Remark 2.12.i), the projections of the six reducible fibres are twelve bitangents of $C_{F,c}$ forming a distinguished Steiner hexad $\mathbf{H}$. Moreover, this Steiner hexad is clearly Galois invariant. When the 28 bitangents of $C_{F,c}$ are equipped with a numbering, distinguishing the Steiner hexad $\mathbf{H}$, then the above considerations apply. They show that the operation of $\operatorname{Gal}(l/k)$ on the twelve bitangents forming $\mathbf{H}$ agrees with the one described by i, but only up to conjugation by an element of $S_2 \wr S_6$.

Remarks 4.2. i) This is not entirely adequate to our situation, as only subgroups of $(S_2 \wr S_6) \cap A_{12}$ may occur. Even worse, as discussed in Remark 1.3.i), two numberings on the bitangents of a plane quartic that distinguish the same Steiner hexad may differ only by the conjugation with an element of $U_{63} \cong (S_2 \wr S_6) \cap A_{12}$.

ii) Thus, it may happen that Proposition 3.5 provides a quartic that behaves in an unwanted way. The operation of $\operatorname{Gal}(l/k)$ on the twelve bitangents of $C_{F,c}$, forming the distinguished Steiner hexad $\mathbf{H}$, might differ from the desired one, described by i. The difference would then be given by the outer automorphism Ψ of $(S_2 \wr S_6) \cap A_{12}$, provided by the conjugation with an element from $(S_2 \wr S_6)\backslash A_{12}$. Fortunately, in this case, $C_{F,-c}$ behaves well, as the next Proposition shows.

Convention. We adopt here the usual convention that Ψ is determined only up to inner automorphisms.

Remark 4.3. An experiment shows that for some but not all subgroups

$$g \subseteq (S_2 \wr S_6) \cap A_{12} \cong U_{63} \subset G\,,$$

the groups g and $\Psi(g)$ are conjugate in G.

Proposition 4.4. *Let k be a field and $F(T) = T^6 + a_5T^5 + \cdots + a_1T + a_0 \in k[T]$ be a separable, monic polynomial of degree six, where $a_0 = c^2$, for some $c \in k\backslash\{0\}$.*

a) *Then the two conic bundles*

$$\operatorname{pr}\colon B_{F,c} \to \mathbf{P}^1 \qquad \textit{and} \qquad \operatorname{pr}\colon B_{F,-c} \to \mathbf{P}^1$$

both split over the splitting field l of $F(T^2)$.

b) *The operations of* $\operatorname{Gal}(l/k)$ *on the components of the reducible fibres, however, differ by the outer automorphism Ψ of $(S_2 \wr S_6) \cap A_{12}$.*

Proof. a) is clear from Proposition 3.5.b).ii).

b) *First step.* The relative situation.
The construction of $B_{F,c}$, as described in Proposition 3.5, assigns to a monic polynomial $F(T) = T^6 + a_5T^5 + \cdots + a_0$ and a choice of c such that $c^2 = a_0$ a conic bundle with exactly six singular fibres. As noticed in Remark 3.6.ii), the whole process may be carried out in a relative situation, such that the old construction reappears when working fibre-by-fibre.

Concretely, this means the following. The affine space $\mathbf{A}^6_k$ with coordinate functions $a_0, \ldots, a_5$ admits the ramified double cover, given by $w^2 = a_0$, which is again an affine space $\operatorname{Spec} k[w, a_1, \ldots, a_5] \cong \mathbf{A}^6_k$. Over a certain open subscheme $\boldsymbol{W} \subset \operatorname{Spec} k[w, a_1, \ldots, a_5]$ to be specified below, the construction provides a family of conic bundles, which is the hypersurface

$$\boldsymbol{B} \subset \boldsymbol{W} \times \mathbf{P}^1 \times \mathbf{P}^2 ,$$

given by $(T_0\, T_1\, T_2)\, M\, (T_0\, T_1\, T_2)^t = 0$. Here, M is the 3×3-matrix (11).

The equation $\det M = 0$ defines the locus $\boldsymbol{L} \subset \boldsymbol{W} \times \mathbf{P}^1$, over which the conics degenerate to rank two, i.e. to the union of two lines. Degeneration to even lower ranks does not occur. The locus $\boldsymbol{L}$ is, in fact, contained in $\boldsymbol{W} \times \mathbf{A}^1$ and given by the equation $T^6 + a_5T^5 + \cdots + a_1T + w^2 = 0$.

The lines themselves hence form a $\mathbf{P}^1$-bundle over a double cover $\boldsymbol{X}$ of $\boldsymbol{L}$. Over $\boldsymbol{L}$, $\boldsymbol{X}$ is given by the equation $W^2 = \det M_{33}$, for M_{33} the upper left 2×2-minor of M. It turns out that $\det M_{33}$ coincides with T, up to a factor being a perfect square.

Second step. The base scheme.
In the affine space $\mathbf{A}^6_k$ with coordinate functions $a_0, \ldots, a_5$, there is the discriminant locus $\Delta \subset \mathbf{A}^6_k$ of the polynomial $T^{12} + a_5T^{10} + \cdots + a_0$. This is a reducible divisor consisting of two components. One is the hyperplane $Z(a_0)$, the other is the discriminant locus of the polynomial $T^6 + a_5T^5 + \cdots + a_0$.

We specify $\boldsymbol{W}$ to be the preimage of $\mathbf{A}^6_k \backslash \Delta$ under the double cover $w = a_0^2$. In fact, in Proposition 3.5, we only worked with points on the somewhat smaller base scheme $\mathbf{A}^6_k \backslash \Delta \backslash \Delta_{\mathrm{sing}}$, but this will not make any difference for the argument below.

There are the finite étale morphisms, i.e. étale covers,

$$\boldsymbol{X} \stackrel{b}{\longrightarrow} \boldsymbol{W} \stackrel{q}{\longrightarrow} \mathbf{A}^6_k \backslash \Delta$$

of k-schemes. Here, $q\colon \boldsymbol{W} \to \mathbf{A}^6_k \backslash \Delta$ is the structural double cover morphism. Moreover, $b\colon \boldsymbol{X} \to \boldsymbol{W}$ is the composition of the natural projection $\boldsymbol{X} \twoheadrightarrow \boldsymbol{L}$ with the composition

$$\boldsymbol{L} \hookrightarrow \boldsymbol{W} \times \mathbf{P}^1 \stackrel{\mathrm{pr}_1}{\twoheadrightarrow} \boldsymbol{W} .$$

According to its construction, $\boldsymbol{X}$ is isomorphic to the integral closure [EGA II, Section 6.3] of $\boldsymbol{W}$ relative to the extension $Q(\boldsymbol{W})[T]/(T^{12} + a_5T^{10} + \cdots + a_1T^2 + w^2)$ of the function field $Q(\boldsymbol{W})$. Moreover, under this isomorphism, $b\colon \boldsymbol{X} \to \boldsymbol{W}$ goes over into the structural morphism. Since the discriminant locus has been taken out, $b\colon \boldsymbol{X} \to \boldsymbol{W}$ is indeed étale [SGA1, Exp. I, Corollaire 7.4].

The twelve lines occurring in the conic bundle $B_{F,c}$ are in a natural bijection with the twelve geometric points on the corresponding fibre of b. Hence, in what follows it suffices to consider the étale cover $b\colon \boldsymbol{X} \to \boldsymbol{W}$ instead of the lines themselves.

Third step. The generic point.
Over the generic point of $\mathbf{A}^6_k$ with function field $\boldsymbol{F} = k(a_0,\ldots,a_5)$ lies exactly one point of $\boldsymbol{W}$, the generic point η, corresponding to the function field

$$\boldsymbol{F}(\sqrt{a_0}) = k(\sqrt{a_0}, a_1, \ldots, a_5) = k(w, a_1, \ldots, a_5)\,.$$

Above η, there is the generic fibre $\boldsymbol{X}_\eta$, which is again only one point and corresponds to the function field

$$Q(\boldsymbol{X}_\eta) = k(w, a_1, \ldots, a_5)[T]/(T^{12} + a_5T^{10} + \cdots + a_1T^2 + w^2)\,.$$

The Galois group of the polynomial $T^{12}+a_5T^{10}+\cdots+a_1T^2+a_0$ over $\boldsymbol{F}$ is equal to $S_2 \wr S_6$, as the coefficients $a_0,\ldots,a_5$ are indeterminates. Moreover, the discriminant of $T^{12} + a_5T^{10} + \cdots + a_0$ is a_0, up to factors being squares in $\boldsymbol{F}$. Thus,

$$\boldsymbol{F}(\sqrt{a_0}) \subset \operatorname{Split}_{\boldsymbol{F}}(T^{12} + a_5T^{10} + \cdots + a_1T^2 + a_0)\,,$$

which reduces the Galois group over $\boldsymbol{F}(\sqrt{a_0})$ to only even permutations.

As a consequence of this, the Galois hull $\overline{\boldsymbol{X}}$ of $\boldsymbol{X}$ over $\boldsymbol{W}$ is automatically Galois over $\mathbf{A}^6_k\backslash\Delta$. The étale cover

$$q\circ\overline{b}\colon \overline{\boldsymbol{X}} \stackrel{\overline{b}}{\longrightarrow} \boldsymbol{W} \stackrel{q}{\longrightarrow} \mathbf{A}^6_k\backslash\Delta$$

of Galois group $\operatorname{Aut}(\overline{\boldsymbol{X}}) \cong S_2 \wr S_6$ is split into two parts, both of which are étale covers and Galois, with Galois groups $\operatorname{Aut}(\boldsymbol{W}) \cong \mathbb{Z}/2\mathbb{Z}$ for q and $\operatorname{Aut}_{\boldsymbol{W}}(\overline{\boldsymbol{X}}) \cong (S_2 \wr S_6) \cap A_{12}$ for $\overline{b}$.

Fourth step. Étale fundamental groups.
Now let $F \in k[T]$ be a monic polynomial of degree six that is separable and has constant term $a_0 = c^2$. The coefficients of this polynomial define two k-rational points (F,c), $(F,-c) \in \boldsymbol{W}(k)$ lying over the same point $(F,c^2) \in (\mathbf{A}^6_k\backslash\Delta)(k)$.

Let us fix an algebraic closure $\overline{k}$ and geometric points $\overline{(F,c)}\colon \operatorname{Spec}\overline{k} \to \boldsymbol{W}$ and $\overline{(F,-c)}\colon \operatorname{Spec}\overline{k} \to \boldsymbol{W}$, lying above (F,c) and $(F,-c)$, respectively, that are mapped under q to the same geometric point $\overline{(F,c^2)}$ on $\mathbf{A}^6_k\backslash\Delta$. We also fix two geometric points p_c and p_{-c} on $\overline{\boldsymbol{X}}$, which are mapped under $\overline{b}$ to $\overline{(F,c)}$ and $\overline{(F,-c)}$, respectively, and shall serve as base points for the operations of the étale fundamental groups on the étale covers $\overline{b}$ and $q\circ\overline{b}$.

Furthermore, there are the natural homomorphisms

$$\begin{aligned} q_*^{(c)}\colon \pi_1^{\text{ét}}(\boldsymbol{W}, \overline{(F,c)}) &\longrightarrow \pi_1^{\text{ét}}(\mathbf{A}^6_k\backslash\Delta, \overline{(F,c^2)}) \quad \text{and} \\ q_*^{(-c)}\colon \pi_1^{\text{ét}}(\boldsymbol{W}, \overline{(F,-c)}) &\longrightarrow \pi_1^{\text{ét}}(\mathbf{A}^6_k\backslash\Delta, \overline{(F,c^2)}) \end{aligned}$$

between the étale fundamental groups [SGA1, Exp. V], which are injective and onto a normal subgroup of index 2. Indeed, q is an étale double cover and $\boldsymbol{W}$ is connected.

In addition, the étale cover $q \circ \overline{b}$ corresponds, under the equivalence of categories described in [SGA1, Exp. V, Section 7], to a surjective continuous group homomorphism

$$\varrho\colon \pi_1^{\text{ét}}(\mathbf{A}_k^6 \backslash \Delta, \overline{(F, c^2)}) \longrightarrow \operatorname{Aut}(\overline{\boldsymbol{X}}) \,.$$

Similarly, the étale cover $\overline{b}$ yields the surjective continuous homomorphisms

$$\varrho_c\colon \pi_1^{\text{ét}}(\boldsymbol{W}, \overline{(F, c)}) \to \operatorname{Aut}_{\boldsymbol{W}}(\overline{\boldsymbol{X}}) \quad \text{and} \quad \varrho_{-c}\colon \pi_1^{\text{ét}}(\boldsymbol{W}, \overline{(F, -c)}) \to \operatorname{Aut}_{\boldsymbol{W}}(\overline{\boldsymbol{X}}) \,,$$

respectively. Both are compatible with ϱ in the sense that the diagrams

$$\begin{array}{ccc} \pi_1^{\text{ét}}(\boldsymbol{W}, \overline{(F, c)}) & \xrightarrow{\ \varrho_c\ } & \operatorname{Aut}_{\boldsymbol{W}}(\overline{\boldsymbol{X}}) \\ {\scriptstyle q_*^{(c)}} \big\downarrow & & \big\downarrow {\scriptstyle \text{incl.}} \\ \pi_1^{\text{ét}}(\mathbf{A}_k^6 \backslash \Delta, \overline{(F, c^2)}) & \xrightarrow{\ \varrho\ } & \operatorname{Aut}(\overline{\boldsymbol{X}}) \end{array} \quad \text{and} \quad \begin{array}{ccc} \pi_1^{\text{ét}}(\boldsymbol{W}, \overline{(F, -c)}) & \xrightarrow{\ \varrho_{-c}\ } & \operatorname{Aut}_{\boldsymbol{W}}(\overline{\boldsymbol{X}}) \\ {\scriptstyle q_*^{(-c)}} \big\downarrow & & \big\downarrow {\scriptstyle \text{incl.}} \\ \pi_1^{\text{ét}}(\mathbf{A}_k^6 \backslash \Delta, \overline{(F, c^2)}) & \xrightarrow{\ \varrho\ } & \operatorname{Aut}(\overline{\boldsymbol{X}}) \end{array}$$

commute.

Next, we have to match the two operations ϱ_c and ϱ_{-c} against each other. For this, we choose an isomorphism $\iota\colon \Phi_{\overline{(F,-c)}} \to \Phi_{\overline{(F,c)}}$ of fibre functors, i.e. a "homotopy class of paths"

$$s \in \pi_1^{\text{ét}}(\boldsymbol{W}, \overline{(F, c)}, \overline{(F, -c)})$$

in the fundamental groupoid, cf. [SGA1, Exp. V, Section 7] or [De, Paragraph 10.16]. We assume that s lifts to a path in $\pi_1^{\text{ét}}(\overline{\boldsymbol{X}}, p_c, p_{-c})$. The class s defines an isomorphism

$$\begin{array}{rcl} \iota_s\colon \pi_1^{\text{ét}}(\boldsymbol{W}, \overline{(F, c)}) & \longrightarrow & \pi_1^{\text{ét}}(\boldsymbol{W}, \overline{(F, -c)}) \,, \\ \sigma & \mapsto & s \circ \sigma \circ s^{-1}, \end{array}$$

by conjugation.

Fortunately, $q_*(s) \in \pi_1^{\text{ét}}(\mathbf{A}_k^6 \backslash \Delta, \overline{(F, c^2)})$ is an element of an ordinary étale fundamental group. Thus, the commutativity of the diagrams above yields that

$$\begin{aligned} \varrho_{-c}(\iota_s(\sigma)) = \varrho_{-c}(s \circ \sigma \circ s^{-1}) = \varrho\big(q_*^{(-c)}(s \circ \sigma \circ s^{-1})\big) = \varrho\big(q_*(s) \circ q_*^{(-c)}(\sigma) \circ q_*(s)^{-1}\big) = \\ \varrho(q_*(s)) \cdot \varrho(q_*^{(c)}(\sigma)) \cdot \varrho(q_*(s))^{-1} = \varrho(q_*(s)) \cdot \varrho_c(\sigma) \cdot \varrho(q_*(s))^{-1}, \end{aligned}$$

for every $\sigma \in \pi_1^{\text{ét}}(\boldsymbol{W}, \overline{(F, c)})$. I.e., $\varrho_{-c} \circ \iota_s$ and ϱ_c differ by conjugation with $\varrho(q_*(s))$,

$$\varrho_{-c} \circ \iota_s = \varrho(q_*(s)) \cdot \varrho_c \cdot \varrho(q_*(s))^{-1} \,.$$

We note finally that, according to its construction, $q_*(s)$ is an element of the fundamental group $\pi_1^{\text{ét}}(\mathbf{A}_k^6 \backslash \Delta, \overline{(F, c^2)})$ that does not lift to the fundamental group of $\boldsymbol{W}$. Hence, it lies in the only nontrivial coset $\pi_1^{\text{ét}}(\mathbf{A}_k^6 \backslash \Delta, \overline{(F, c^2)}) \setminus q_*^{(c)}\big(\pi_1^{\text{ét}}(\boldsymbol{W}, \overline{(F, c)})\big)$ and one has

$$\varrho(q_*(s)) \in \operatorname{Aut}(\overline{\boldsymbol{X}}) \setminus \operatorname{Aut}_{\boldsymbol{W}}(\overline{\boldsymbol{X}}) \,.$$

In other words, the operations $\varrho_{-c} \circ \iota_s$ and ϱ_c indeed differ by the outer automorphism Ψ of $\operatorname{Aut}_{\boldsymbol{W}}(\overline{\boldsymbol{X}}) \cong (S_2 \wr S_6) \cap A_{12}$.

Fifth step. Conclusion–The absolute Galois group.
In order to complete the proof, we still have to specify which class of paths $s \in \pi_1^{\text{ét}}(\boldsymbol{W}, \overline{(F,c)}, \overline{(F,-c)})$ to work with. For this, let us write k^{sep} for the separable closure of k within the chosen algebraic closure $\overline{k}$. Then the k-rational point $(F,-c)\colon \operatorname{Spec} k \to \boldsymbol{W}$ yields a homomorphism

$$i_{-c}\colon \operatorname{Gal}(k^{\text{sep}}/k) = \pi_1^{\text{ét}}(\operatorname{Spec} k, \operatorname{Spec} \overline{k}) \longrightarrow \pi_1^{\text{ét}}(\boldsymbol{W}, \overline{(F,-c)})\,.$$

Similarly, the k-rational point $(F,c)\colon \operatorname{Spec} k \to \boldsymbol{W}$ provides a homomorphism $i_c\colon \operatorname{Gal}(k^{\text{sep}}/k) = \pi_1^{\text{ét}}(\operatorname{Spec} k, \operatorname{Spec} \overline{k}) \longrightarrow \pi_1^{\text{ét}}(\boldsymbol{W}, \overline{(F,c)})$.

The compositions $\varrho_{-c} \circ i_{-c}$ and $\varrho_c \circ i_c$ then describe the operations of $\operatorname{Gal}(k^{\text{sep}}/k)$ on the étale cover $\overline{b}$, obtained by taking p_{-c} and p_c, respectively, as base points. I.e., the Galois operation on the fibres $\overline{b}^{-1}(\overline{(F,-c)})$ and $\overline{b}^{-1}(\overline{(F,c)})$. Both these fibres are transitive $\operatorname{Gal}(k^{\text{sep}}/k)$-sets isomorphic to

$$\operatorname{Gal}(k^{\text{sep}}/k)/\operatorname{Gal}(k^{\text{sep}}/\operatorname{Split}_k(F(T^2)))\,.$$

Taking p_{-c} and p_c as generators, we thus find a bijection $\overline{b}^{-1}(\overline{(F,-c)}) \cong \overline{b}^{-1}(\overline{(F,c)})$ that is compatible with the Galois operations. This may be extended to an isomorphism between the fibre functors, i.e. to a class $s \in \pi_1^{\text{ét}}(\boldsymbol{W}, \overline{(F,c)}, \overline{(F,-c)})$, as required.

By construction, the homomorphisms $i_{-c}, \iota_s \circ i_c\colon \operatorname{Gal}(k^{\text{sep}}/k) \to \pi_1^{\text{ét}}(\boldsymbol{W}, \overline{(F,-c)})$ agree modulo elements of $\pi_1^{\text{ét}}(\boldsymbol{W}, \overline{(F,-c)})$ acting trivially on $\overline{\boldsymbol{X}}$. Thus, the result of the previous step implies that

$$\varrho_{-c} \circ \iota_s \circ i_c = \varrho_{-c} \circ i_{-c}\colon \operatorname{Gal}(k^{\text{sep}}/k) \to \operatorname{Aut}_{\boldsymbol{W}}(\overline{\boldsymbol{X}}) \text{ and } \varrho_c \circ i_c\colon \operatorname{Gal}(k^{\text{sep}}/k) \to \operatorname{Aut}_{\boldsymbol{W}}(\overline{\boldsymbol{X}})$$

differ by the outer automorphism Ψ of $\operatorname{Aut}_{\boldsymbol{W}}(\overline{\boldsymbol{X}}) \cong (S_2 \wr S_6) \cap A_{12}$, as claimed. □

Theorem 4.5. *Let an infinite field k of characteristic not 2, a normal and separable extension field l, and an injective group homomorphism*

$$i\colon \operatorname{Gal}(l/k) \hookrightarrow U_{63} \subset G \cong \operatorname{Sp}_6(\mathbb{F}_2) \subset S_{28}$$

be given. Then there exists a nonsingular quartic curve C over k such that l is the field of definition of the 28 bitangents and each $\sigma \in \operatorname{Gal}(l/k)$ permutes the bitangents as described by $i(\sigma) \in G \subset S_{28}$.

Proof. There is an isomorphism $U_{63} \cong (S_2 \wr S_6) \cap A_{12}$, which describes the operation of U_{63} on the twelve lines of the distinguished Steiner hexad.

We are thus given an injection $i'\colon \operatorname{Gal}(l/k) \hookrightarrow (S_2 \wr S_6) \cap A_{12}$, to which Lemma 3.3 can be applied. It yields a polynomial $F = T^6 + a_5T^5 + \cdots + a_1T + a_0 \in k[T]$ of degree six, such that the splitting field of $F(T^2)$ is exactly l. More explicitly, $l = k(\sqrt{A_1}, ..., \sqrt{A_6})$, for $A_1, \ldots, A_6$ the roots of F. Furthermore, the operation of $\operatorname{Gal}(l/k)$ on the square roots $\pm\sqrt{A_i}$ agrees with the natural operation of $S_2 \wr S_6$ on twelve objects forming six pairs. As discussed in Corollary 3.4, there is a Zariski dense subset of $k[T]_6$ of polynomials with the same behaviour.

As $\operatorname{Gal}(l/k) \overset{i'}{\hookrightarrow} A_{12}$, we know that the discriminant of $F(T^2)$ is a square in k. In other words, $a_0 \in (k^*)^2$. Hence, we may apply Proposition 3.5 in order to find two nonsingular plane quartics $C_{F,c}$ and $C_{F,-c}$. According to Lemma 2.11, they both enjoy the following property. Twelve of their bitangents, which form Steiner hexads $\mathbf{H}^+$ and $\mathbf{H}^-$, respectively, are defined over l and permuted by $\mathrm{Gal}(l/k)$ exactly in the way described by the given injection $\mathrm{Gal}(l/k) \stackrel{i'}{\hookrightarrow} S_2 \wr S_6$.

Let us consider $C_{F,c}$ first. Having equipped the 28 bitangents of $C_{F,c}$ with any numbering distinguishing the Steiner hexad $\mathbf{H}^+$, the operation of $\mathrm{Gal}(l/k)$ on $\mathbf{H}^+$ agrees with the desired one up to an inner automorphism φ of $S_2 \wr S_6$. This might be the conjugation with an even element. In this case, φ extends to an inner automorphism of the whole of G, which completes the argument, as discussed in Remark 1.3.i). Otherwise, Proposition 4.4 shows that $C_{F,-c}$ has all the properties required. ☐

Remark 4.6 (The case that k is a number field). Let k be a number field and g a subgroup of $G \cong \mathrm{Sp}_6(\mathbb{F}_2)$ that is contained in U_{63}. Then there exists a nonsingular quartic curve C over k such that the natural permutation representation

$$i\colon \mathrm{Gal}(\overline{k}/k) \longrightarrow G \subset S_{28}$$

on the 28 bitangents of C has the subgroup g as its image.

Proof. According to Theorem 4.5, it suffices to show that, for every number field k and each subgroup $g \subseteq U_{63}$, there exists a normal extension field l such that $\mathrm{Gal}(l/k)$ is isomorphic to g. This is a particular instance of the inverse Galois problem, but the groups occurring are easy enough.

In fact, among the 1155 conjugacy classes of subgroups of G that are contained in U_{63}, 1119 consist of solvable groups. For these, the inverse Galois problem has been solved by I. R. Shafarevich [Sha], cf. [NSW, Theorem 9.5.1]. The groups in the remaining 36 conjugacy classes all turn out to be factor groups of the wreath product $S_2 \wr H$, where $H \subset S_{15}$ is isomorphic to A_5, S_5, A_6, or S_6, cases for which Galois extensions are known over an arbitrary number field [KM, paragraph 2.2.3]. ☐

Example 4.7. Put

$$F := T^6 - 24T^5 + 152T^4 - 340T^3 + 335T^2 - 150T + 25 \in \mathbb{Q}[T]\,.$$

Then the construction described in Proposition 3.5 yields the plane quartics

$$\begin{aligned}-2T_0^3T_2 + 37T_0^2T_1T_2 + 67T_0^2T_2^2 + 2T_0T_1^3 - 10T_0T_1^2T_2 + 114T_0T_1T_2^2 + 166T_0T_2^3 + 4T_1^4 \\ - 168T_1^3T_2 + 42T_1^2T_2^2 + 369T_1T_2^3 - 45T_2^4 = 0\end{aligned}$$

for $c = 5$ and

$$\begin{aligned}-2T_0^3T_2 + 43T_0^2T_1T_2 + 480T_0^2T_2^2 + 2T_0T_1^3 - 36T_0T_1^2T_2 - 2460T_0T_1T_2^2 + 8700T_0T_2^3 \\ + 2T_1^4 - 189T_1^3T_2 + 5170T_1^2T_2^2 - 31255T_1T_2^3 + 40250T_2^4 = 0\end{aligned}$$

for $c = -5$. In this case, the two subgroups of G occurring as the Galois groups operating on the 28 bitangents are in fact conjugate to each other.

The Galois group of the polynomial F itself is A_4, realised as a subgroup of S_6 by the operation on 2-sets. On the other hand, the Galois group of $F(T^2)$ is of index 8 in $(S_2 \wr A_4) \cap A_{12}$, of order 48. According to the classification of transitive groups in degree twelve, due to G. F. Royle [Ro] (cf. [CHM]) and used by `magma` as well as `gap`, it corresponds to number 12T31.

Example 4.8. Over the function field $\mathbb{F}_3(t)$, the polynomial

$$T^6 + (2t^4 + 2t^2)T^4 + (2t^6 + 2t^4 + t^2 + 1)T^3 + (t^8 + t^6 + 2t^4 + 2t^2)T^2 + (2t^{10} + 2t^4)T + t^{12} + 2t^6 + 1 \in \mathbb{F}_3(t)[T]$$

provides the same Galois group. We obtain the nonsingular plane quartics over $\mathbb{F}_3(t)$, given by

$$\begin{aligned}&T_0^4 + 2T_0^3T_1 + (2t^6+t^4+t^2)T_0^3T_2 + (t^4+t^2+1)T_0^2T_1T_2 + (2t^4+t^2+2)T_0^2T_2^2\\&+ T_0T_1^2T_2 + (2t^6+2t^4+2)T_0T_1T_2^2 + (t^{10}+t^8+2t^4+2)T_0T_2^3 + T_1^3T_2\\&\qquad + (2t^4+2)T_1^2T_2^2 + (t^8+t^6+t^2+1)T_1T_2^3 + 2t^2T_2^4 = 0\end{aligned}$$

and

$$\begin{aligned}&T_0^4 + 2T_0^3T_1 + (t^6+t^4+t^2+2)T_0^3T_2 + (t^4+t^2+1)T_0^2T_1T_2\\&+ (2t^{10}+2t^8+2t^6+t^4+1)T_0^2T_2^2 + T_0T_1^2T_2 + (2t^6+2t^4+2)T_0T_1T_2^2\\&+ (t^{12}+2t^{10}+2t^2+1)T_0T_2^3 + T_1^3T_2 + (2t^6+2t^4+1)T_1^2T_2^2\\&+ (t^{10}+2t^8+2t^6+t^4+2t^2+2)T_1T_2^3 + (2t^{16}+2t^{12}+t^{10}+t^6+2t^4+2t^2+2)T_2^4 = 0\,.\end{aligned}$$

Example 4.9. Put

$$F := T^6 - 3T^5 - 2T^4 + 9T^3 - 5T^1 + 1 \in \mathbb{Q}[T]\,.$$

Then the construction described in Proposition 3.5 yields the plane quartics

$$-T_0^3T_2 - 2T_0^2T_1T_2 + 14T_0^2T_2^2 - 2T_0T_1^3 + 9T_0T_1^2T_2 + 4T_0T_1T_2^2 - 7T_0T_2^3 + T_1^4 - 2T_1^3T_2 - 7T_1^2T_2^2 + 3T_2^4 = 0$$

for $c = 1$ and

$$-T_0^3T_2 - 2T_0^2T_1T_2 + 6T_0^2T_2^2 - 2T_0T_1^3 + 9T_0T_1^2T_2 + 12T_0T_1T_2^2 - 3T_0T_2^3 + T_1^4 + 6T_1^3T_2 + 5T_1^2T_2^2 - 4T_1T_2^3 - T_2^4 = 0$$

for $c = -1$. In this case, the two subgroups of G occurring as the Galois groups on the 28 bitangents are not conjugate to each other. For example, the second is contained in the subgroup $U_{36} \subset G$, too, while the first one is not.

The Galois group of the polynomial F itself is S_3, realised as a subgroup of S_6 by the regular representation. The Galois group of $F(T^2)$ is of index 2 in $(S_2 \wr S_3) \cap A_{12}$, of order 96. According to the classification of transitive groups in degree twelve, it is number 12T69.

Example 4.10. Over the function field $\mathbb{F}_3(t)$, the polynomial

$$T^6 + 2t^2T^5 + t^4T^4 + (2t^7 + t^4 + t^2 + 1)(T^3 + t^2T^2) + (t^7 + t^4 + t^2 + 1)^2 \in \mathbb{F}_3(t)[T]$$

provides the same Galois group. We obtain the nonsingular plane quartics over $\mathbb{F}_3(t)$, given by

$$\begin{aligned}&T_0^4 + 2T_0^3T_1 + (2t^7+2t^4)T_0^3T_2 + (t^2+1)T_0^2T_1T_2 + (t^9+2t^7+2t^4+2t^2+2)T_0^2T_2^2\\ &+ (2t^2+1)T_0T_1^2T_2 + (2t^6+t^4+2)T_0T_1T_2^2 + (2t^{11}+2t^9+t^7+t^6+t^4+t^2+2)T_0T_2^3\\ &+ T_1^3T_2 + (t^2+2)T_1^2T_2^2 + (t^9+t^7+t^6+2t^4+2t^2+1)T_1T_2^3\\ &\qquad + (t^{14}+2t^{13}+t^{12}+t^{11}+2t^9+2t^6+t^4)T_2^4 = 0\end{aligned}$$

and

$$\begin{aligned}&T_0^4 + 2T_0^3T_1 + (t^7+t^4+2t^2+2)T_0^3T_2 + (t^2+1)T_0^2T_1T_2 + (t^7+2t^6+1)T_0^2T_2^2\\ &+ (2t^2+1)T_0T_1^2T_2 + (t^9+2t^4+t^2+2)T_0T_1T_2^2\\ &+ (t^{14}+2t^{11}+2t^9+2t^8+t^7+2t^6+t^4+2t^2+1)T_0T_2^3 + T_1^3T_2 + (2t^7+2t^4+1)T_1^2T_2^2\\ &+ (2t^9+2t^7+2t^6+t^4+t^2+2)T_1T_2^3\\ &\qquad + (t^{18}+2t^{14}+2t^{13}+t^{11}+t^{10}+2t^9+2t^8+t^6+t^2+2)T_2^4 = 0\,.\end{aligned}$$

Remark 4.11. The examples above were chosen from the enormous supply in the hope that they are of some particular interest. The Galois groups realised are in fact the minimal ones that yield the generic orbit type [12, 16] on the 28 bitangents. The corresponding conic bundle surfaces are of the minimal possible Picard rank 2, and their generic quadratic twists are of Picard rank 1.

5. Twisting

There is the double cover $p\colon \widetilde{G} \to G$ of finite groups, for $W(E_7) \cong \widetilde{G} \subset S_{56}$ and $\mathrm{Sp}_6(\mathbb{F}_2) \cong G \subset S_{28}$, which is given by the operation on the size two blocks. The kernel of p is exactly the centre $Z \subset \widetilde{G}$. For a subgroup $H \subset \widetilde{G}$, one therefore has two options.

i) Either $p|_H\colon H \to p(H)$ is two-to-one. Then $H = p^{-1}(h)$, for $h := p(H)$. In this case, H contains the centre of $\widetilde{G}$ and, as abstract groups, one has an isomorphism $H \cong p(H) \times \mathbb{Z}/2\mathbb{Z}$.

ii) Or $p|_H\colon H \to p(H)$ is bijective.

In our geometric setting, the first case is the generic one. More precisely, let $C\colon q = 0$ be a nonsingular plane quartic such that the 28 bitangents are acted upon by the group $h \subseteq G$. Then, for λ an indeterminate, the 56 exceptional curves on $S_\lambda\colon \lambda w^2 = q$ operated upon by $h \times \mathbb{Z}/2\mathbb{Z}$.

Lemma 5.1. *Let k be any field and $C\colon q = 0$ a nonsingular plane quartic over k. Write l_0 for the field of definition of the 28 bitangents of C and let $h \subset G$ be the subgroup, via which* $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ *operates on them.*

Furthermore, let $S_t\colon tw^2 = q$ be the universal double cover of $\mathbf{P}^2_{k(t)}$, ramified at C, defined over the function field $k(t)$. Then the following holds.

a) *The Galois group* $\mathrm{Gal}(k(t)^{\mathrm{sep}}/k(t))$ *operates on the 56 exceptional curves of S_t via $p^{-1}(h)$.*

b) *The field of definition of the 56 exceptional curves of* S_t *is* $\boldsymbol{l} = l_0(t)(\sqrt{ct})$, *for some* $c \in k^*$.

Proof. a) One only has to show that the field $\boldsymbol{l}$ of definition of the 56 exceptional curves is strictly larger than the composite $l_0 \cdot k(t) = l_0(t)$. For this, let us recall the local formula (4), which shows that, indeed, $\boldsymbol{l} = l_0(t)(\sqrt{ct})$, for a certain $c \in l_0^*$.

b) The field $\boldsymbol{l}$ is necessarily Galois over $k(t)$ and, as we are in case i), the natural exact sequence

$$0 \longrightarrow \mathrm{Gal}(\boldsymbol{l}/l_0(t)) \longrightarrow \mathrm{Gal}(\boldsymbol{l}/k(t)) \xrightarrow{\mathrm{res}} \mathrm{Gal}(l_0(t)/k(t)) \longrightarrow 0$$

is split. I.e.,

$$\mathrm{Gal}(\boldsymbol{l}/k(t)) \cong \mathrm{Gal}(l_0(t)/k(t)) \times \mathbb{Z}/2\mathbb{Z} = \mathrm{Gal}(l_0/k) \times \mathbb{Z}/2\mathbb{Z}\,.$$

This shows that $\boldsymbol{l} = l_0(t)(\sqrt{p(t)})$, for some polynomial $p(t) \in k[t]$. On the other hand, we just found that $\boldsymbol{l} = l_0(t)(\sqrt{ct})$, for a certain constant $c \in l_0^*$. Both results may be true, simultaneously, only if $\boldsymbol{l} = l_0(t)(\sqrt{ct})$, where c is in k^*. □

For particular choices of λ, every subgroup of $\widetilde{G}$ may be realised that has image h under the projection p.

Theorem 5.2. *Let a field* k *of characteristic not* 2, *a normal and separable extension field* l, *and an injective group homomorphism* $i\colon \mathrm{Gal}(l/k) \hookrightarrow \widetilde{G}$ *be given. Write* l_0 *for the subfield corresponding to* $i^{-1}(Z)$ *under the Galois correspondence.*

a) *Then there is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Gal}(l/k) & \overset{i}{\hookrightarrow} & \widetilde{G} \\ {\scriptstyle \mathrm{res}}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{Gal}(l_0/k) & \overset{\bar{\imath}}{\hookrightarrow} & G\,, \end{array}$$

the downward arrow on the left being the restriction.

b) *Let* $C\colon q = 0$ *be a nonsingular plane quartic over* k, *the 28 bitangents of which are defined over* l_0 *and acted upon by* $\mathrm{Gal}(l_0/k)$ *as described by* $\bar{\imath}$. *Then there exists some* $\lambda \in k^*$ *such that the 56 exceptional curves of the degree two del Pezzo surface*

$$S_\lambda\colon \lambda w^2 = q$$

are defined over l *and each automorphism* $\sigma \in \mathrm{Gal}(l/k)$ *permutes them as described by* $i(\sigma) \in \widetilde{G} \subset S_{56}$.

Proof. a) follows directly from Galois theory.

b) Consider the universal double cover S_t over $k(t)$, given by $tw^2 = q$. According to Lemma 5.1, this belongs to the first of the two cases distinguished above. Moreover, the field $\boldsymbol{l}$ of definition of the 56 exceptional curves is $\boldsymbol{l} = l_0(t)(\sqrt{ct})$, for some $c \in k^*$. Let us once again distinguish between the two cases, as above.

First case. $\mathrm{Gal}(l/k) \cong \mathrm{Gal}(l_0/k) \times \mathbb{Z}/2\mathbb{Z}$.
Then, according to Kummer theory, $l = l_0(\sqrt{a})$ for some $a \in k^*$. Moreover $\sqrt{a} \notin l_0$. Specialising t to $\lambda := ac$, we find exactly the required field of definition.

Second case. $l = l_0$.
In this case, a subgroup $H \subset \mathrm{Gal}(\boldsymbol{l}/k(t))$ of index two has been chosen that, under restriction, is mapped isomorphically onto $\mathrm{Gal}(l_0(t)/k(t))$. We want to specialise t in such a way that we find H as the Galois group over k.

For this, we first observe that, under the Galois correspondence, the subgroup H corresponds to a quadratic extension field $\boldsymbol{q}$ of $k(t)$. Thus, one has $H = \mathrm{Gal}(\boldsymbol{l}/\boldsymbol{q})$, which yields that $\boldsymbol{q} \not\subseteq l_0(t)$. Indeed, otherwise, $H = \mathrm{Gal}(\boldsymbol{l}/\boldsymbol{q}) \supseteq \mathrm{Gal}(\boldsymbol{l}/l_0(t))$, but $\mathrm{Gal}(\boldsymbol{l}/l_0(t)) \cong \mathbb{Z}/2\mathbb{Z}$ is annihilated under res.

On the other hand, according to Galois theory, the field extension $\boldsymbol{l}/k(t)$, i.e. $l_0(t)(\sqrt{ct})/k(t)$, has exactly three types of quadratic intermediate fields. These are
i) the extension fields of the type $k(t)(\sqrt{b})$, for $k(\sqrt{b})$ a quadratic subfield of l_0,
ii) the extension field $k(t)(\sqrt{ct})$,
iii) the extension fields of the type $k(t)(\sqrt{b{\cdot}ct})$, for $k(\sqrt{b})$ a quadratic subfield of l_0.
Among these, type i) is excluded to us, as these fields are contained in $l_0(t)$. Thus, the subgroup H is realised as the Galois group over an intermediate field $k(t)(\sqrt{b't})$, for some constant $b' \in k^*$. Specialising t to $\lambda := b'$ yields the subgroup H over k. $\square$

Corollary 5.3. *Let an infinite field k of characteristic not 2, a normal and separable extension field l, and an injective group homomorphism*
$$i\colon \mathrm{Gal}(l/k) \hookrightarrow p^{-1}(U_{63})$$
be given. Then there exists a degree two del Pezzo surface S over k such that l is the field of definition of the 56 exceptional curves and each $\sigma \in \mathrm{Gal}(l/k)$ permutes them as described by $i(\sigma) \in \widetilde{G} \subset S_{56}$.

Proof. This follows from Theorem 5.2 together with Theorem 4.5. $\square$

Remark 5.4. Let $C\colon q^2 - q_1q_2 = 0$ (cf. Proposition 2.3.a)) be a nonsingular plane quartic over a field k with a Galois invariant Steiner hexad and $\lambda \in k^*$ a non-square.
i) Then the two conic bundles, associated with the Steiner hexad, on the twist $S_\lambda\colon \lambda w^2 = q^2 - q_1q_2$ are no longer k-rational, but defined over $k(\sqrt{\lambda})$ and conjugate to each other.
ii) On the other hand, in this case, S_λ carries a global Brauer class $\alpha \in \mathrm{Br}(S_\lambda)_2$, which, over the function field $k(S_\lambda)$, is given by the quaternion algebra $(\lambda, \frac{q_1}{T_0^2})_2$, for T_0 one of the coordinate functions on $\mathbf{P}^2$.

Proof. According to [Ma, Lemma 43.1.1 and Proposition 31.3], we only have to show that $\mathrm{div}(\frac{q_1}{T_0^2})$ is the norm of a divisor on $(S_\lambda)_{k(\sqrt{\lambda})}$. The equation
$$(q - \sqrt{\lambda}w)(q + \sqrt{\lambda}w) = q_1q_2$$
shows that $Z(q - \sqrt{\lambda}w, q_1) - Z(T_0)$ indeed is such a divisor on $(S_\lambda)_{k(\sqrt{\lambda})}$. $\square$

The 2-torsion Brauer classes on degree two del Pezzo surfaces have been systematically studied by P. Corn in [Co]. It turns out that the conjugacy classes of subgroups of $W(E_7)$ that lead to such a Brauer class form a partially ordered set with exactly two maximal elements. The subgroup $p^{-1}(U_{63})$ studied here is one of them. It yields the Brauer classes of the first type in Corn's terminology.

6. An application: Cubic surfaces with a Galois invariant double-six

A nonsingular cubic surface over an algebraically closed field contains exactly 27 lines. The maximal subgroup $G_{\max} \subset S_{27}$ that respects the intersection pairing is isomorphic to the Weil group $W(E_6)$ [Ma, Theorem 23.9] of order 51 840.

A double-six (cf. [Ha, Remark V.4.9.1] or [Do, Subsection 9.1.1]) is a configuration of twelve lines $E_1, \ldots, E_6, E'_1, \ldots, E'_6$ such that

i) $E_1, \ldots, E_6$ are mutually skew,

ii) $E'_1, \ldots, E'_6$ are mutually skew, and

iii) $E_i \cdot E'_j = 1$, for $i \neq j$, $1 \leqslant i, j \leqslant 6$, and $E_i \cdot E'_i = 0$ for $i = 1, \ldots, 6$.

Every cubic surface contains exactly 36 double-sixes, which are transitively acted upon by $G_{\max}$ [Do, Theorem 9.1.3]. Thus, there is an index-36 subgroup $U_{\rm ds} \subset G_{\max}$ stabilising a double-six. This is one of the maximal subgroups of $G_{\max} \cong W(E_6)$. Up to conjugation, $W(E_6)$ has maximal subgroups of indices 2, 27, 36, 40, 40, and 45.

As an application of Theorem 5.2, we have the following result.

Theorem 6.1. *Let an infinite field k of characteristic not 2, a normal and separable extension field l, and an injective group homomorphism*

$$i\colon \operatorname{Gal}(l/k) \hookrightarrow U_{\rm ds} \subset G_{\max} \cong W(E_6) \subset S_{27}$$

be given. Then there exists a nonsingular cubic surface S over k such that

i) *the 27 lines on S are defined over l and each $\sigma \in \operatorname{Gal}(l/k)$ permutes them as described by $i(\sigma) \in U_{\rm ds} \subset S_{27}$.*

ii) *S is k-unirational.*

Proof. There is an injective homomorphism $\iota\colon W(E_6) \hookrightarrow W(E_7)$ that corresponds to the blow-up of a point on a cubic surface. The subgroup $\iota(U_{\rm ds}) \subset W(E_7)$ is of index $36{\cdot}56 = 2016$. It is sufficient to show that the image $\overline{\iota}(U_{\rm ds})$ under the composition

$$\overline{\iota}\colon W(E_6) \stackrel{\iota}{\hookrightarrow} W(E_7) \stackrel{p}{\twoheadrightarrow} W(E_7)/Z \cong G$$

is contained in U_{63}. Indeed, then Corollary 5.3 yields a degree two del Pezzo surface S' of degree two with a k-rational line L and the proposed Galois operation on the 27 lines that do not meet L. Blowing down L, we obtain a nonsingular cubic surface S satisfying i). As there is a k-rational point on S, the blow down of L, S is k-unirational [Ko02, Theorem 2].

To show the group-theoretic claim, let us consider a model cubic surface over an algebraically closed field. The group $U_{\rm ds}$ operates on the 27 lines with an orbit type $[12, 15]$, the orbit of size twelve being a double-six [EJ10, Example 9.2]. Hence, $\iota(U_{\rm ds})$

operates on the 56 exceptional curves of the degree two del Pezzo surface, obtained by blowing up one point, with orbit type $[1, 1, 12, 12, 15, 15]$.

A size twelve orbit consists of exceptional curves $E_1, \ldots, E_6, E'_1, \ldots, E'_6$ such that

i) $E_1, \ldots, E_6$ are mutually skew,

ii) $E'_1, \ldots, E'_6$ are mutually skew, and

iii) $E_i \cdot E'_j = 1$, for $i \neq j$, $1 \leqslant i, j \leqslant 6$, and $E_i \cdot E'_i = 0$ for $i = 1, \ldots, 6$.

The second orbit $\{\widetilde{E}_1, \ldots, \widetilde{E}_6, \widetilde{E}'_1, \ldots, \widetilde{E}'_6\}$ of size twelve is obtained from the first by applying the Geiser involution g.

Let us now consider the auxiliary set $\{E_1, \ldots, E_6, \widetilde{E}'_1, \ldots, \widetilde{E}'_6\}$ of exceptional curves. Then, according to formula (9),

i) $E_1, \ldots, E_6$ are mutually skew,

ii) $\widetilde{E}'_1, \ldots, \widetilde{E}'_6$ are mutually skew, and

iii) $E_i \cdot \widetilde{E}'_j = 0$, for $i \neq j$, $1 \leqslant i, j \leqslant 6$, and $E_i \cdot \widetilde{E}'_i = 1$ for $i = 1, \ldots, 6$.

Thus, Lemma 2.11 shows that

$$\{(\pi(E_1), \pi(\widetilde{E}'_1)), \ldots, (\pi(E_6), \pi(\widetilde{E}'_6))\} = \{(\pi(E_1), \pi(E'_1)), \ldots, (\pi(E_6), \pi(E'_6))\}$$

is a Steiner hexad. In particular, the image $\overline{\iota}(U_{\mathrm{ds}})$ stabilises a Steiner hexad and is hence contained in U_{63}, as claimed. □

Remark 6.2. The group $W(E_6)$ has 350 conjugacy classes of subgroups. Among these, 102 stabilise a double-six, i.e. are contained in U_{ds}, so that Theorem 6.1 applies. For each such conjugacy class, we constructed an example of a of cubic surface over $\mathbb{Q}$ using a different method [EJ10].

7. Another application: Cubic surfaces with a rational line

The maximal subgroup $U_{\mathrm{l}} \subset G_{\max}$ of index 27 is just the stabiliser of a line. We have the following application to cubic surfaces with a rational line, which seems to be a slight refinement of the results given in [KST, Section 6].

Theorem 7.1. *Let an infinite field k of characteristic not 2, a normal and separable extension field l, and an injective group homomorphism*

$$i\colon \operatorname{Gal}(l/k) \hookrightarrow U_{\mathrm{l}} \subset G_{\max} \cong W(E_6) \subset S_{27}$$

be given. Then there exists a nonsingular cubic surface S over k such that

i) *the 27 lines on S are defined over l and each $\sigma \in \operatorname{Gal}(l/k)$ permutes them as described by $i(\sigma) \in U_{\mathrm{ds}} \subset S_{27}$.*

ii) *S is k-unirational.*

Proof. As above, it suffices to show that $\overline{\iota}(U_{\mathrm{l}})$ is contained in U_{63}. For this, once again, we consider a model cubic surface over an algebraically closed field. The group U_{l} operates on the 27 lines with an orbit type $[1, 10, 16]$, the orbit of size ten consisting of the lines that intersect the invariant line L. Hence, $\iota(U_{\mathrm{l}})$ operates on the 56 exceptional curves of the degree two del Pezzo surface, obtained by blowing up one point, with orbit type $[1, 1, 1, 1, 10, 10, 16, 16]$.

It is well known [Be, Lemma IV.15] that the exceptional curves in any of the size ten orbits may be written in the form $E_1, \ldots, E_5, E'_1, \ldots, E'_5$, where

i) $E_1, \ldots, E_5$ are mutually skew,

ii) $E'_1, \ldots, E'_5$ are mutually skew, and

iii) $E_i \cdot E'_j = 0$, for $i \neq j$, $1 \leqslant i, j \leqslant 5$, and $E_i \cdot E'_i = 1$ for $i = 1, \ldots, 5$.

In addition, the image E_0 of the invariant line fulfils $E_0 \cdot E_i = E_0 \cdot E'_i = 1$, for $1, \ldots, 5$. Finally, the inverse image E of the blow-up point is skew to all these curves.

Formula (9) now shows that $E_1, \ldots, E_5, E, E'_1, \ldots, E'_5, \widetilde{E}_0$ fulfil, in this order, the assumptions of Lemma 2.11. In particular, the image $\overline{\iota}(U_l)$ stabilises a Steiner hexad and is hence contained in U_{63}, as required. □

Corollary 7.2. *Let an infinite field k of characteristic not 2, a normal and separable extension field l, and an injective group homomorphism*

$$i\colon \operatorname{Gal}(l/k) \hookrightarrow g_{\max} \cong W(D_5) \subset S_{16}$$

be given. Then there exists a del Pezzo surface D of degree four over k such that

i) *the 16 exceptional curves on D are defined over l and each $\sigma \in \operatorname{Gal}(l/k)$ permutes them as described by $i(\sigma) \in g_{\max} \subset S_{16}$.*

ii) *D is k-unirational.* □

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