

Moduli spaces and the inverse Galois problem for cubic surfaces

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joint work with
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The geometry of smooth cubic surfaces

Let $C \subseteq \mathbf{P}^3$ be a smooth cubic surface over an algebraically closed field. Classical algebraic geometry gives a lot of information about such surfaces. For instance,

- C is isomorphic to \mathbf{P}^2 , blown up in six points in general position.
- C contains precisely 27 lines.

The configuration of the 27 lines is highly symmetric. The group of all permutations respecting the intersection pairing is isomorphic to the Weyl group $W(E_6)$ of order 51 840.

- There is a pentahedron associated with a general C (Sylvester).
- There are (at least) two kinds of moduli spaces, coming out of classical invariant theory.
 - The coarse moduli scheme of smooth cubic surfaces (Salmon, Clebsch).
 - The fine moduli scheme of *marked cubic surfaces* (Cayley, Coble).

An arithmetic problem

Suppose the base field is \mathbb{Q} . Then the 27 lines on C are defined only over $\overline{\mathbb{Q}}$. $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ permutes them. The intersection pairing must be respected. I.e., after having fixed a marking on the lines, there is a homomorphism $\rho: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow W(E_6)$.

Definition

One says that the Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts upon the lines of C via $G := \text{im } \rho \subseteq W(E_6)$.

When no marking is chosen, the subgroup G is determined only up to conjugation.

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Problem

Let $G \subseteq W(E_6)$ be any subgroup. Is there a smooth cubic surface C over \mathbb{Q} such that $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts via G upon the lines of C ?

The main result

Theorem (Elsenhans+J.)

Let \mathfrak{g} be an arbitrary conjugacy class of subgroups of $W(E_6)$.

Then there exists a smooth cubic surface C over \mathbb{Q} such that $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the lines of C via a subgroup $G \subseteq W(E_6)$ belonging to the class \mathfrak{g} .

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Remark

There are 350 conjugacy classes of subgroups in $W(E_6)$.

The result is proven by a list of 350 examples.

These are available on our web pages, e.g.

<http://www.uni-math.gwdg.de/jahnel/linkstopapers.html>.

Actually, we offer six lists.

How to verify the lists

This is the problem the other way round. Given a cubic surface such as

$$g_1 := 5x^3 - 9x^2y + 15x^2z - 216x^2w + 2xy^2z + 5xy^2w - 255xz^2 - 96xz^2w + 104xw^2 + 9y^2z - 4y^2w + 11yz^2 - 139yz^2w - 16yw^2 - 45z^3 + 35z^2w - 11zw^2 + 108w^3;$$

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Algorithm (Computation of the 27 lines and the Galois group)

- 1 Parametrise a line ℓ in the form $w = ay + bz, x = cy + dz$. The condition that $\ell \subset C$ means

$$(a:c:1:0), (b:d:0:1), ((a+b):(c+d):1:1), ((a-b):(c-d):1:(-1)) \in C.$$

This defines an ideal $I \subset \mathbb{Q}[a, b, c, d]$, generated by four polynomials, which are at most cubic.

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- 2 Calculate a Gröbner base of I . Typically, it will consist of a univariate polynomial $p \in \mathbb{Q}[a]$ of degree 27 and formulas to compute b, c , and d from a .
Otherwise, apply a random automorphism of \mathbf{P}^3 and go back to step 1.

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Otherwise, apply a random automorphism of \mathbf{P}^3 and go back to step 1.
- 3 Calculate the Galois group of p . (Stauduhar's algorithm, Cannon-Holt)
Output a list of generating permutations and the intersection matrix.

Experiments

In a sample of 20000 random surfaces with coefficients in $[0 \dots 50]$, we found the whole $W(E_6)$, 20000 times.

The same for 20000 random surfaces with coefficients in $[-100 \dots 100]$.

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Remark

There are several very interesting constructions that work in particular cases. [Explicit blow up of six points on \mathbf{P}^2 , Hexahedral form, Cayley-Salmon form, each combined with explicit Galois descent]

Nevertheless, the plan of this talk is to ignore about these and to present a construction that, in principle, works for an arbitrary subgroup $G \subseteq W(E_6)$.

The pentahedron

Definition (Sylvester)

The cubic surface $S^{(a_0, \dots, a_4)}$ given in \mathbf{P}^4 by

$$\begin{aligned} a_0 X_0^3 + a_1 X_1^3 + a_2 X_2^3 + a_3 X_3^3 + a_4 X_4^3 &= 0, \\ X_0 + X_1 + X_2 + X_3 + X_4 &= 0, \end{aligned}$$

is said to be in *pentahedral form*.

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Remarks

- The five planes given by $X_k = 0$ for $k = 0, \dots, 4$ form the *pentahedron* associated with the cubic surface.
- Over an algebraically closed field, a general cubic surface has a unique pentahedron.
- If $a_0, \dots, a_4 \neq 0$ then the Hessian [covariant] of $S^{(a_0, \dots, a_4)}$ is a quartic surface having exactly ten singular points. These are the triple intersection points of the five planes.

The idea of explicit Galois descent

Suppose that a_0, \dots, a_4 are not in \mathbb{Q} , but the zeroes of a separable polynomial $f \in \mathbb{Q}[T]$ of degree 5. Then $S^{(a_0, \dots, a_4)}$ is, a priori, defined only over the splitting field L of f .

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A construction

- Assume that f is separable. Then $L' := \mathbb{Q}[T]/(f)$ is an étale algebra having exactly five homomorphisms $i_0, \dots, i_4: L' \rightarrow L$ [embeddings if L' is a field]. For $a = \overline{T} \in L'$, one has $a_k = i_k(a)$, $k = 0, \dots, 4$.

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- Choose a linear form $l \in L'[w, x, y, z]$ such that $l^{i_0} + \dots + l^{i_4} = 0$, but no further linear relations are true.

(This is fulfilled when the four coefficients of l form a basis of the L' -vector space defined by $\text{tr } x = 0$.)

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- Choose a linear form $l \in L'[w, x, y, z]$ such that $l^{i_0} + \dots + l^{i_4} = 0$, but no further linear relations are true.
(This is fulfilled when the four coefficients of l form a basis of the L' -vector space defined by $\text{tr } x = 0$.)
- Then the cubic surface $S_{(a_0, \dots, a_4)}$ over \mathbb{Q} defined by

$$\text{Tr } al^3 = 0$$

is L -isomorphic to $S^{(a_0, \dots, a_4)}$.

Remarks

- This construction immediately yields an algorithm. Computations have to be done in L' , not in the Galois hull L .
- Substituting linear forms into the coordinates was a standard technique of the 19th century geometers.

The idea of explicit Galois descent II

Remarks

- This construction immediately yields an algorithm. Computations have to be done in L' , not in the Galois hull L .
- Substituting linear forms into the coordinates was a standard technique of the 19th century geometers.
- There is the general concept of Galois descent due to A. Weil, which was further generalised by A. Grothendieck. This requires a *descent datum* $\{U_\sigma\}_{\sigma \in G}$ for $G := \text{Gal}(L/\mathbb{Q})$.

Here, $U_\sigma: S^{(a_0, \dots, a_4)} \rightarrow S^{(a_0, \dots, a_4)}$ has to be a morphism twisted by σ and the U_σ together have to form a group operation.

- The construction above yields Galois descent for $U_\sigma = T_\sigma \circ \sigma$, where

$$\sigma: (x_0 : \dots : x_4) \mapsto (\sigma(x_0) : \dots : \sigma(x_4)),$$

$$T_\sigma: (x_0 : \dots : x_4) \mapsto (x_{\Pi(\sigma)^{-1}(0)} : \dots : x_{\Pi(\sigma)^{-1}(4)}),$$

and $\Pi: G \rightarrow S_5$ is the permutation representation defined by the operation of G on a_0, \dots, a_4 .

Definition

Let S be any scheme. Then a *family of cubic surfaces* over S or a *cubic surface* over S is a flat morphism $p: C \rightarrow S$ such that there exist

- a rank-4 vector bundle \mathcal{E} on S ,
- a non-zero section $c \in \Gamma(\mathcal{O}(3), \mathbf{P}(\mathcal{E}))$, and
- an isomorphism $\text{div}(c) \xrightarrow{\cong} C$ of S -schemes.

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Definitions

- 1 A *line* on a smooth cubic surface $p: C \rightarrow S$ is a \mathbf{P}^1 -bundle $l \subset C$ over S such that, for every $x \in S$, one has $\deg_{\mathcal{O}(1)} l_x = 1$.

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- 2 A *family of marked cubic surfaces* over S or a *marked cubic surface* over S is a cubic surface $p: C \rightarrow S$ together with a sequence (l_1, \dots, l_6) of six mutually disjoint lines.

The sequence (l_1, \dots, l_6) itself is called a *marking* on C .

Theorem

Let K be a field.

- 1 Then there exists a fine moduli scheme $\tilde{\mathcal{M}}$ for marked cubic surfaces over K . I.e., the functor

$$F: \{K\text{-schemes}\} \longrightarrow \{\text{sets}\},$$
$$S \mapsto \{\text{marked cubic surfaces over } S\} / \sim$$

is representable by a K -scheme $\tilde{\mathcal{M}}$.

- 2 $\tilde{\mathcal{M}}$ is a smooth, quasi-projective fourfold and, in addition, a rational variety.

The moduli scheme of marked cubic surfaces

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Idea of proof: Let $\mathcal{U} \subset (\mathbf{P}^2)^6$ be the open subscheme parametrising all ordered 6-tuples of points in \mathbf{P}^2 in general position. By the Hilbert-Mumford numerical criterion, \mathcal{U} is contained in the PGL_3 -stable locus. One proves that $\mathcal{M} := \mathcal{U} / \mathrm{PGL}_3$ represents the functor F .

The moduli scheme of marked cubic surfaces II

To give a point $p \in \mathcal{U}(\overline{K})$ is equivalent to giving $p_1, \dots, p_6 \in \mathbf{P}^2(\overline{K})$ in general position. Projective geometry shows that there is a unique $\gamma \in \mathrm{PGL}_3(\overline{K})$ that maps (p_1, p_2, p_3, p_4) to the standard projective basis $((1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : 1 : 1))$.

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Observation (A naive embedding)

The \overline{K} -rational points on $\widetilde{\mathcal{M}}$ may thus be represented by 3×6 -matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & 1 & w & y \\ 0 & 1 & 0 & 1 & x & z \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Vanishing of the third coordinate of p_5 would mean that p_1, p_2 , and p_5 were collinear, and similarly for p_6 .

The moduli scheme of marked cubic surfaces III

Corollary

We have an open embedding $\widetilde{\mathcal{M}} \hookrightarrow \mathbf{A}^4$. In fact, $\widetilde{\mathcal{M}}$ is an affine scheme.

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Remark

The moduli scheme $\widetilde{\mathcal{M}}$ of marked cubic surfaces has its origins in the work of Arthur Cayley from 1849. His approach was as follows.

- There are 45 tritangent planes meeting the surface in three lines. Through each line there are five tritangent planes.
- This leads to a total of 135 cross ratios, which are invariants of the cubic surface, as soon as a marking is fixed on the lines. Only 45 of these cross ratios are essentially different, due to constraints within the cubic surfaces.
- They provide an embedding $\widetilde{\mathcal{M}} \hookrightarrow (\mathbf{P}^1)^{45}$. The closure of the image is Cayley's "cross ratio variety".

Cayley's "cross ratio variety" was rediscovered by I. Naruki in 1982.

Coble's compactification—The gamma variety

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SL_3 operates naturally on $\mathcal{O}(n)$, and hence on $\Gamma(\mathbf{P}^2, \mathcal{O}(n))$, for every n . But PGL_3 does not. There is no PGL_3 -linearisation for $\mathcal{O}_{\mathbf{P}^2}(1)$.

Via the canonical isogeny $SL_3 \twoheadrightarrow PGL_3$, the kernel of which consists of the multiples of the identity matrix by the third roots of unity, there is a canonical PGL_3 -linearisation for $\mathcal{O}(3)$.

Further, the PGL_3 -invariant sections are the same as the SL_3 -invariant ones.

Minors of the 6×3 -matrix

$$\begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} \\ x_{2,0} & x_{2,1} & x_{2,2} \\ \dots & & \\ x_{6,0} & x_{6,1} & x_{6,2} \end{pmatrix}$$

define invariant sections on $(\mathbf{P}^2)^6$.

Notation

- For example, for $1 \leq i_1 < i_2 < i_3 \leq 6$,

$$m_{i_1, i_2, i_3} := \det \begin{pmatrix} x_{i_1, 0} & x_{i_1, 1} & x_{i_1, 2} \\ x_{i_2, 0} & x_{i_2, 1} & x_{i_2, 2} \\ x_{i_3, 0} & x_{i_3, 1} & x_{i_3, 2} \end{pmatrix}$$

defines an invariant section of $\mathcal{O}(n_1) \boxtimes \dots \boxtimes \mathcal{O}(n_6)$. Here, $n_i := 1$ when $i \in \{i_1, i_2, i_3\}$ and $n_i := 0$, otherwise.

We write $m_{i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}} := m_{i_1, i_2, i_3}$ for $\sigma \in S_3$.

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We write $m_{i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}} := m_{i_1, i_2, i_3}$ for $\sigma \in S_3$.

- Further,

$$d_2 := \det \begin{pmatrix} x_{1,0}^2 & x_{1,1}^2 & x_{1,2}^2 & x_{1,0}x_{1,1} & x_{1,0}x_{1,2} & x_{1,1}x_{1,2} \\ x_{2,0}^2 & x_{2,1}^2 & x_{2,2}^2 & x_{2,0}x_{2,1} & x_{2,0}x_{2,2} & x_{2,1}x_{2,2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x_{6,0}^2 & x_{6,1}^2 & x_{6,2}^2 & x_{6,0}x_{6,1} & x_{6,0}x_{6,2} & x_{6,1}x_{6,2} \end{pmatrix}$$

is an invariant section of $\mathcal{O}(2) \boxtimes \dots \boxtimes \mathcal{O}(2)$.

Remark

One has the beautiful relation

$$d_2 = -\det \begin{pmatrix} m_{1,3,4} m_{1,5,6} & m_{1,3,5} m_{1,4,6} \\ m_{2,3,4} m_{2,5,6} & m_{2,3,5} m_{2,4,6} \end{pmatrix}.$$

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Definition (Coble 1917)

For $\{i_1, \dots, i_6\} = \{1, \dots, 6\}$, consider

$$\begin{aligned} \gamma_{(i_1 i_2 i_3)(i_4 i_5 i_6)} &:= m_{i_1, i_2, i_3} m_{i_4, i_5, i_6} d_2 && \text{and} \\ \gamma_{(i_1 i_2)(i_3 i_4)(i_5 i_6)} &:= m_{i_1, i_3, i_4} m_{i_2, i_3, i_4} m_{i_3, i_5, i_6} m_{i_4, i_5, i_6} m_{i_5, i_1, i_2} m_{i_6, i_1, i_2} \end{aligned}$$

Following the original work, we call these 40 SL_3 -invariant, and hence PGL_3 -invariant, sections $\gamma \in \Gamma((\mathbf{P}^2)^6, \mathcal{O}(3) \boxtimes \dots \boxtimes \mathcal{O}(3))$ the *irrational invariants*.

Remarks

- Within the parentheses, the indices may be arbitrarily permuted without changing the symbol.

Further, in the symbols $\gamma_{(i_1 i_2 i_3)(i_4 i_5 i_6)}$, the two triples may be interchanged.

However, in the symbols $\gamma_{(i_1 i_2)(i_3 i_4)(i_5 i_6)}$, the three pairs may be permuted only cyclically. Thus, altogether there are ten invariants of the first type and 30 invariants of the second type.

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- The irrational invariants γ_i do *not* generate the invariant ring

$$\bigoplus_{d \geq 0} \Gamma((\mathbf{P}^2)^6, \mathcal{O}(3d) \boxtimes \dots \boxtimes \mathcal{O}(3d))^{\mathrm{PGL}_3}$$

and do *not* define an embedding of the categorical quotient $((\mathbf{P}^2)^6)^{\mathrm{semi-stable}} / \mathrm{PGL}_3$ into \mathbf{P}^{39} .

They behave well, however, on $\mathcal{U} / \mathrm{PGL}_3 \subset ((\mathbf{P}^2)^6)^{\mathrm{semi-stable}} / \mathrm{PGL}_3$.

Coble's compactification—The gamma variety V

Notation

The PGL_3 -invariant local sections of $\mathcal{O}(3) \boxtimes \dots \boxtimes \mathcal{O}(3)$ form an invertible sheaf on $\widetilde{\mathcal{M}} = \mathcal{U} / \mathrm{PGL}_3$, which we will denote by \mathcal{L} .

Theorem (Coble)

- 1 The invertible sheaf \mathcal{L} on $\widetilde{\mathcal{M}}$ is very ample.
- 2 The 40 irrational invariants $\gamma_i \in \Gamma(\widetilde{\mathcal{M}}, \mathcal{L})$ define a projective embedding $\gamma: \widetilde{\mathcal{M}} \hookrightarrow \mathbf{P}_K^{39}$.
- 3 The Zariski closure of the image of γ is contained in a nine-dimensional linear subspace.
- 4 In this \mathbf{P}^9 , it is the intersection of 30 cubic hypersurfaces.

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Theorem (Coble)

- 1 The invertible sheaf \mathcal{L} on $\widetilde{\mathcal{M}}$ is very ample.
- 2 The 40 irrational invariants $\gamma \in \Gamma(\widetilde{\mathcal{M}}, \mathcal{L})$ define a projective embedding $\gamma: \widetilde{\mathcal{M}} \hookrightarrow \mathbf{P}_K^{39}$.
- 3 The Zariski closure of the image of γ is contained in a nine-dimensional linear subspace.
- 4 In this \mathbf{P}^9 , it is the intersection of 30 cubic hypersurfaces.

Definitions

- We call $\gamma: \widetilde{\mathcal{M}} \hookrightarrow \mathbf{P}_K^{39}$ Coble's gamma map.
- The Zariski closure of the image of γ is called Coble's gamma variety and denoted by \widetilde{M} .

Remarks

- We prove part 2, first. This implies part 1.
- To prove 2, we work in the naive embedding into \mathbf{A}^4 . The gamma map is then given by 40 explicit polynomials. We show that they separate points and tangent vectors. (This requires some computer work.)

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- Part 3 is equivalent to the statement that the vector space $\langle \gamma. \rangle$ is only of dimension ten. This was known to Coble in 1917 and is easily checked by computer.
- Finally, the purely cubic expressions in the $\gamma.$ form a vector space of dimension 190. Since $\binom{12}{3} = 220$, this shows that there are 30 cubic relations, except for those coming from the linear ones.

The intersection of the 30 cubic hypersurfaces in \mathbf{P}^9 is reported by `magma` as being reduced and irreducible of dimension four.

The operation of $W(E_6)$

A marked cubic surface over S is automatically smooth, according to our definition. All its 27 lines are defined over S . They may be labelled as $l_1, \dots, l_6, l'_1, \dots, l'_6, l''_{12}, l''_{13}, \dots, l''_{56}$.

There are exactly 51 840 possible markings for a smooth cubic surface with all 27 lines defined over the base. They are acted upon, transitively, by a group of that order, isomorphic to the Weyl group $W(E_6)$.

Convention

We identify $W(E_6)$ with the permutation group acting on the 27 labels $l_1, \dots, l_6, l'_1, \dots, l'_6, l''_{12}, l''_{13}, \dots, l''_{56}$.

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$W(E_6)$ naturally operates on the moduli functor

$$F: \{K\text{-schemes}\} \rightarrow \{\text{sets}\} .$$

Therefore, it operates on the moduli scheme $\tilde{\mathcal{M}}$.

The operation of $W(E_6)$ II

Coble's (as well as Cayley's) compactifications explicitly linearise the operation of $W(E_6)$.

Lemma (Coble)

There exists a $W(E_6)$ -linearisation of $\mathcal{L} \in \text{Pic}(\tilde{\mathcal{M}})$ such that

- *the 80 sections $\pm\gamma \in \Gamma(\tilde{\mathcal{M}}, \mathcal{L})$ form a $W(E_6)$ -invariant set.*
- *The corresponding permutation representation $\Pi: W(E_6) \hookrightarrow S_{80}$ is transitive. It has a system of 40 blocks given by the pairs $\{\gamma, -\gamma\}$.*
- *The permutation representation $W(E_6) \hookrightarrow S_{40}$ on the 40 blocks is the same as that on decompositions of the 27 lines into three pairs of Steiner trihedra.*

It is also defined by the operation of $W(E_6)$ on its cosets modulo one of its maximal subgroups of index 40.

The operation of $W(E_6)$ III

Idea of the proof: We need a system of compatible isomorphisms $i_g: T_g^* \mathcal{L} \xrightarrow{\cong} \mathcal{L}$ for $T_g: \widetilde{\mathcal{M}} \rightarrow \widetilde{\mathcal{M}}$ the operation of g .

For $g \in S_6 \subset W(E_6)$, there is an obvious such isomorphism defined by the permutation of the six labels.

$W(E_6)$ is generated by S_6 and one additional element, the quadratic transformation h_{123} . In the naive coordinates, this map is given by $(w, x, y, z) \mapsto (\frac{1}{w}, \frac{1}{x}, \frac{1}{y}, \frac{1}{z})$.

List explicit formulas for the 40 irrational invariants γ_i in terms of these coordinates. Plugging in $(w, x, y, z) \mapsto (\frac{1}{w}, \frac{1}{x}, \frac{1}{y}, \frac{1}{z})$ in a naive way, yields an isomorphism $i'_{h_{123}}: T_{h_{123}}^* \mathcal{L} \xrightarrow{\cong} \mathcal{L}$, permuting the 40 sections γ_i up to signs and a common scaling factor of $\frac{1}{w^2 x^2 y^2 z^2}$. \square

Un-marked cubic surfaces

As cubic surfaces may have automorphisms, a fine moduli scheme cannot exist.

Facts

- The quotient $\widetilde{\mathcal{M}} / W(E_6) =: \mathcal{M}$ is the coarse moduli scheme of smooth cubic surfaces.
- On the other hand, $\mathcal{M} \cong \mathcal{V} / \mathrm{PGL}_4$ for $\mathcal{V} \subset \mathbf{P}^{19}$ the open subscheme parametrising smooth cubic surfaces.

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Remarks

- Every smooth cubic surface corresponds to a PGL_4 -stable point in \mathbf{P}^{19} .
- The PGL_4 -invariants have been determined by A. Clebsch as early as 1861. In today's language, Clebsch's result is that there is an open embedding $\mathrm{Cl}: \mathcal{V}/\mathrm{PGL}_4 \cong \mathcal{M} \hookrightarrow \mathbf{P}(1, 2, 3, 4, 5)$.
One writes A, \dots, E for the coordinates of $\mathbf{P}(1, 2, 3, 4, 5)$.

Example

The *pentahedral* family $\mathcal{C} \rightarrow \mathbf{P}^4/S_5$ of cubic surfaces is given by

$$\begin{aligned}a_0X_0^3 + a_1X_1^3 + a_2X_2^3 + a_3X_3^3 + a_4X_4^3 &= 0, \\X_0 + X_1 + X_2 + X_3 + X_4 &= 0.\end{aligned}$$

The pentahedron revisited

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Remarks

- 1 One has $\mathbf{P}^4/S_5 \cong \mathbf{P}(1, 2, 3, 4, 5)$. The elementary symmetric functions $\sigma_1, \dots, \sigma_5$ in a_0, \dots, a_4 are natural homogeneous coordinates.
- 2 We restrict our considerations to the open subset $\mathcal{P} \subset \mathbf{P}^4/S_5$ representing smooth cubic surfaces having a *proper pentahedron*. This means $\sigma_5 \neq 0$.

The pentahedron revisited II

Theorem (Clebsch/Salmon)

For $t: \mathcal{P} \rightarrow \mathcal{M}$ the classifying morphism, the composition

$$\text{Clot}: \mathcal{P} \rightarrow \mathcal{M} \hookrightarrow \mathbf{P}(1, 2, 3, 4, 5)$$

is given by the S_5 -invariant sections

$$l_8 := \sigma_4^2 - 4\sigma_3\sigma_5, \quad l_{16} := \sigma_1\sigma_5^3, \quad l_{24} := \sigma_4\sigma_5^4, \quad l_{32} := \sigma_2\sigma_5^6, \quad l_{40} := \sigma_5^8$$

of $\mathcal{O}(8)$, $\mathcal{O}(16)$, $\mathcal{O}(24)$, $\mathcal{O}(32)$, and $\mathcal{O}(40)$, respectively.

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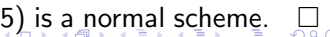
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Lemma

The classifying morphism $t: \mathcal{P} \rightarrow \mathcal{M}$ is an open embedding.

Idea of the proof: This is not a deep observation. One shows actually that $\text{Clot}: \mathcal{P} \rightarrow \mathbf{P}(1, 2, 3, 4, 5)$ is an open embedding. For that, one verifies that Clot is birational and finite, and that $\mathbf{P}(1, 2, 3, 4, 5)$ is a normal scheme. 

Theorem (Eisenhans+J.)

- 1 *The canonical morphism*

$$\psi: \widetilde{\mathcal{M}} \xrightarrow{\text{pr}} \mathcal{M} \xrightarrow{\text{Cl}} \mathbf{P}(1, 2, 3, 4, 5)$$

allows an extension to \mathbf{P}^{39} under the gamma map. More precisely, there exists a rational map $\tilde{\psi}: \mathbf{P}^{39} \dashrightarrow \mathbf{P}(1, 2, 3, 4, 5)$ such that the following diagram commutes,

$$\begin{array}{ccccc} \widetilde{\mathcal{M}} & \xrightarrow{\text{pr}} & \mathcal{M} & \xrightarrow{\text{Cl}} & \mathbf{P}(1, 2, 3, 4, 5) \\ \downarrow \gamma & & & & \parallel \\ \mathbf{P}^{39} & \dashrightarrow & \tilde{\psi} & \dashrightarrow & \mathbf{P}(1, 2, 3, 4, 5). \end{array}$$

Theorem (Elsenhans+J., continued)

② Explicitly, the rational map $\tilde{\psi}: \mathbf{P}^{39} \dashrightarrow \mathbf{P}(1, 2, 3, 4, 5)$, defined by the global sections

- $-6P_2 \in \Gamma(\mathbf{P}^{39}, \mathcal{O}(2))$,
- $-24P_4 + \frac{41}{16}P_2^2 \in \Gamma(\mathbf{P}^{39}, \mathcal{O}(4))$,
- $\frac{576}{13}P_6 - \frac{396}{13}P_4P_2 + \frac{29}{13}P_2^3 \in \Gamma(\mathbf{P}^{39}, \mathcal{O}(6))$,
- $-\frac{62208}{1171}P_8 + \frac{54864}{1171}P_6P_2 + \frac{203616}{1171}P_4^2 - \frac{61287}{1171}P_4P_2^2 + \frac{13393}{4684}P_2^4 \in \Gamma(\mathbf{P}^{39}, \mathcal{O}(8))$,
- $\frac{41472}{155}P_{10} - \frac{4605984}{36301}P_8P_2 - \frac{106272}{403}P_6P_4 + \frac{19990440}{471913}P_6P_2^2 + \frac{47719206}{471913}P_4^2P_2$
 $- \frac{7468023}{471913}P_4P_2^3 + \frac{10108327}{18876520}P_2^5 \in \Gamma(\mathbf{P}^{39}, \mathcal{O}(10))$,

satisfies this condition. Here, P_k denotes the sum of the 40 k -th powers.

③ In other words, these formulas express Clebsch's invariants A, \dots, E in terms of Coble's 40 irrational invariants γ .

Coble's gammas versus Clebsch's invariants III

Idea of the proof: $\psi := \text{Cl} \circ \text{pr}$ defines a rational map

$$\varphi: \tilde{M} \dashrightarrow \mathbf{P}(1, 2, 3, 4, 5)$$

from the gamma variety. Extend ψ to a morphism by closing the graph,

$$\begin{array}{ccc} \mathfrak{M} & & \\ \pi_1 \downarrow & \searrow \varphi' & \\ \tilde{M} & \dashrightarrow \varphi & \mathbf{P}(1, 2, 3, 4, 5). \end{array}$$

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One may show that $\varphi'^* \mathcal{O}(1) \cong \pi_1^* \mathcal{O}(2)|_{\tilde{M}} \otimes \mathcal{O}(-E_1)$, where E_1 is an effective Cartier divisor supported in the exceptional fibres of π_1 . Consequently,

$$\pi_{1*} \varphi'^* \mathcal{O}(i) \subseteq \mathcal{O}(2i)|_{\tilde{M}}.$$

The rational map $\varphi: \tilde{M} \dashrightarrow \mathbf{P}(1, 2, 3, 4, 5)$ is therefore given by sections t_i of $\mathcal{O}(2i)|_{\tilde{M}}$, $i = 1, \dots, 5$.

Coble's gammas versus Clebsch's invariants IV

It is classically known that

$$\varphi'^{-1}(A) = (-6) \sum_{j=0}^{39} X_j^2.$$

The other four sections extend to $\mathcal{O}(4), \dots, \mathcal{O}(10)$, i.e. to \mathbf{P}^{39} .

Indeed, the Castelnuovo-Mumford regularity of $\mathcal{I}_{\tilde{M}}$ may be computed to 5, which shows

$$H^1(\mathbf{P}^{39}, \mathcal{I}_{\tilde{M}}(i)) = 0$$

for $i \geq 4$.

Coble's gammas versus Clebsch's invariants V

The sections t_i may be assumed $W(E_6)$ -invariant. Molien's formula shows

$$\dim \Gamma(\mathbf{P}(V), \mathcal{O}(2i))^{W(E_6)} = \begin{cases} 1 & \text{for } i=1, \\ 2 & \text{for } i=2, \\ 5 & \text{for } i=3, \\ 11 & \text{for } i=4, \\ 23 & \text{for } i=5, \end{cases}$$

for V the relevant 10-dimensional representation of $W(E_6)$.

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All these sections may be found explicitly. The reduction process modulo a Gröbner base of $\mathcal{I}_{\tilde{M}}$ then shows that

$$\Gamma(\tilde{M}, \mathcal{O}(2i)|_{\tilde{M}})^{W(E_6)} = \begin{cases} \langle P_2 \rangle & \text{for } i=1, \\ \langle P_4, P_2^2 \rangle & \text{for } i=2, \\ \langle P_6, P_4 P_2, P_2^3 \rangle & \text{for } i=3, \\ \langle P_8, P_6 P_2, P_4^2, P_4 P_2^2, P_2^4 \rangle & \text{for } i=4, \\ \langle P_{10}, P_8 P_2, P_6 P_4, P_6 P_2^2, P_4^2 P_2, P_4 P_2^3, P_2^5 \rangle & \text{for } i=5. \end{cases}$$

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To determine the coefficients in this basis is, finally, an interpolation problem.

Algorithm (Pentahedron from cubic surface—generic case)

- 1 Determine a Gröbner basis for the ideal $\mathcal{I}_{H_{\text{sing}}} \subset K[X_0, \dots, X_3]$ of the singular locus of the Hessian H of C . In particular, this yields a univariate degree-10 polynomial \bar{F} defining an S_5 -extension.
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- 4 Determine, in a second Gröbner base calculation, an element of minimal degree in the ideal $(\mathcal{I}_{H_{\text{sing}}}, \bar{F}_2) \subset L[X_0, \dots, X_3]$. The result is a linear polynomial l . Its conjugates define the five individual planes that form the pentahedron.
- 5 Scale l by a suitable non-zero factor from L such that $\text{Tr}_{L/K} l = 0$. Then calculate $a \in L$ such that the equation of the surface is exactly $\text{Tr}_{L/K} al^3 = 0$.
- 6 Return a . One might want to return l as a second value.

Remark

The calculation of the pentahedron for a concrete cubic surface was considered a hopeless problem before the emergence of modern computer algebra.

Twisting Coble's gamma variety

Fix a continuous homomorphism $\rho: \text{Gal}(\overline{K}/K) \rightarrow W(E_6)$ and consider

$$F_\rho: \{K\text{-schemes}\} \longrightarrow \{\text{sets}\},$$

$$S \mapsto \left\{ \begin{array}{l} \text{marked cubic surfaces over } S_{\overline{K}} \text{ such that} \\ \text{Gal}(\overline{K}/K) \text{ operates on the 27 lines as} \\ \text{described by } \rho \end{array} \right\} / \sim,$$

the moduli functor, *twisted by* ρ .

Theorem (Elsenhans+J.)

The functor F_ρ is representable by a K -scheme $\widetilde{\mathcal{M}}_\rho$ that is a twist of $\widetilde{\mathcal{M}}$.

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Strategy (to construct a cubic surface for $G \subseteq W(E_6)$)

- 1 First, one should find a Galois extension L/\mathbb{Q} such that $\text{Gal}(L/\mathbb{Q}) \cong G$. This defines the homomorphism ρ .
- 2 Then a \mathbb{Q} -rational point $P \in \widetilde{\mathcal{M}}_\rho(\mathbb{Q})$ is sought for.
- 3 For the corresponding cubic surface \mathcal{C}_P over \mathbb{Q} , the Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ operates on the 27 lines exactly as desired.

The main algorithm

Algorithm (Cubic surface for a given group)

Given a subgroup $G \subseteq W(E_6)$ and a field such that $\text{Gal}(L/\mathbb{Q}) \cong G$, this algorithm computes a smooth cubic surface C over \mathbb{Q} such that $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ operates upon the lines of C via the group $\text{Gal}(L/\mathbb{Q})$.

- 1 Fix a system $\Gamma \subseteq G$ of generators of G . For every $g \in \Gamma$, store the permutation $\Pi(g) \in S_{80}$, which describes the operation of g on the 80 irrational invariants $\pm\gamma$. Further fix, once and for ever, ten of the $\pm\gamma$ that are linearly independent. Express the other 70 explicitly as linear combinations of these basis vectors.

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- 2 For every $g \in \Gamma$, determine the 10×10 -matrix describing the operation of g on the 10-dimensional L -vector space $\langle \gamma \rangle$. Use the explicit basis, fixed in step 1.

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- 2 For every $g \in \Gamma$, determine the 10×10 -matrix describing the operation of g on the 10-dimensional L -vector space $\langle \gamma \rangle$. Use the explicit basis, fixed in step 1.
- 3 Choose an explicit basis of the field L as a \mathbb{Q} -vector space. Finally, make explicit the isomorphism $\rho^{-1}: G \rightarrow \text{Gal}(L/\mathbb{Q}) \subseteq \text{Hom}_{\mathbb{Q}}(L, L)$. I.e., write down a matrix for every $g \in \Gamma$.

The main algorithm II

- ④ (Explicit Galois descent I) The condition that

$$(\sigma(x_{\Pi(\rho(\sigma))^{-1}(0)}), \dots, \sigma(x_{\Pi(\rho(\sigma))^{-1}(79)})) = (x_0, \dots, x_{79})$$

for all $\sigma \in G$ is a \mathbb{Q} -linear system of equations in $10[L : \mathbb{Q}]$ variables. We start with Γ instead of G and get $80[L : \mathbb{Q}] \# \Gamma$ equations. The result is a ten dimensional \mathbb{Q} -vector space $V \subset \langle \gamma. \rangle$, described by an explicit basis.

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Convert the 30 cubic forms defining the image of $\gamma_L: \widetilde{\mathcal{M}}_L \hookrightarrow \mathbf{P}_L^{79}$ into terms of this basis of V . The result are 30 explicit cubic forms with coefficients in \mathbb{Q} . They describe the Zariski closure of $\widetilde{\mathcal{M}}_\rho$ in $\mathbf{P}(V)$.

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- 6 Search for a \mathbb{Q} -rational point on this variety.

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for all $\sigma \in G$ is a \mathbb{Q} -linear system of equations in $10[L : \mathbb{Q}]$ variables. We start with Γ instead of G and get $80[L : \mathbb{Q}] \# \Gamma$ equations. The result is a ten dimensional \mathbb{Q} -vector space $V \subset \langle \gamma. \rangle$, described by an explicit basis.

- 5 (Explicit Galois descent II)

Convert the 30 cubic forms defining the image of $\gamma_L: \widetilde{\mathcal{M}}_L \hookrightarrow \mathbf{P}_L^{79}$ into terms of this basis of V . The result are 30 explicit cubic forms with coefficients in \mathbb{Q} . They describe the Zariski closure of $\widetilde{\mathcal{M}}_\rho$ in $\mathbf{P}(V)$.

- 6 Search for a \mathbb{Q} -rational point on this variety.
- 7 From the coordinates of the point found, read the 40 irrational invariants $\gamma.$. Calculate Clebsch's invariants A, \dots, E from these. Then determine pentahedral coefficients and an explicit equation over \mathbb{Q} .

Summary

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- The main algorithm had to be run only for the seven most complicated conjugacy classes [group orders form 9 to 108].

Thank you!!