Real multiplication on $K3$ surfaces via period integration

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joint work with
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$K3$ surfaces

Definition (abstract definition – classification of algebraic surfaces)

A $K3$ *surface* is a simply connected, projective algebraic surface having a global (algebraic, holomorphic) 2-form without zeroes or poles.

Facts

For X a $K3$ surface, $\pi_1(X,.) = 0$ and $H_2(X,\mathbb{Z}) \cong H^2(X,\mathbb{Z}) \cong \mathbb{Z}^{22}$.

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Examples

1. A smooth quartic in $\mathbf{P}^3$.
2. A double cover of $\mathbf{P}^2$, ramified at a smooth sextic curve.

   Sextics with ordinary double points may be taken, too. Then the resolution of singularities is a $K3$ surface.

In this talk, we work with $K3$ surfaces that are double covers of $\mathbf{P}^2$, ramified over six lines in $\mathbf{P}^2$.

Point counting – An experiment

Consider a "random" example and a very particular one

$$X_1 \ : \ w^2 = x^6 + 2y^6 + 3z^6 + 5x^2y^4 + 7xy^2z^3 + 3y^5z + x^3z^3$$
$$X_2 \ : \ w^2 = (-y^2 + 8yz - 8z^2)(7x^2 + 40xz + 56z^2)(2x^2 + 3xy + y^2)\,.$$

p	$(\#X_1(\mathbb{F}_p) \bmod p)$	$(\#X_2(\mathbb{F}_p) \bmod p)$
23	19	18
29	7	1
31	7	7
37	0	1
41	7	1
43	5	1
47	11	19
53	47	1
59	28	1
61	44	1
67	54	1
71	23	34
73	11	0
79	41	27
83	57	1
89	46	3
97	28	52

Point counting – An experiment II

Observations

1. In the "random" example X_1, there is no regularity to be seen.
2. In example X_2, however, we observe that

$$\#X_2(\mathbb{F}_p) \equiv 1 \pmod{p}$$

for all primes $p \equiv 3, 5 \pmod{8}$.

Remarks

- One also has $\#X_2(\mathbb{F}_{41}) \equiv 1 \pmod{41}$, which is purely accidental.
- The bound of 100 is just for the presentation, one may easily extend it, at least up to 1000.
- The primes $p \equiv 3, 5 \pmod{8}$ are exactly those that are inert in $\mathbb{Q}(\sqrt{2})$.

Recall from the theory of elliptic curves

Fact (An arithmetic consequence of CM)

Let X be an elliptic curve with complex multiplication (CM) by $E = \mathbb{Q}(\sqrt{d})$. Then $\#X(\mathbb{F}_p) \equiv 1 \pmod{p}$ for every prime p that is inert in E.

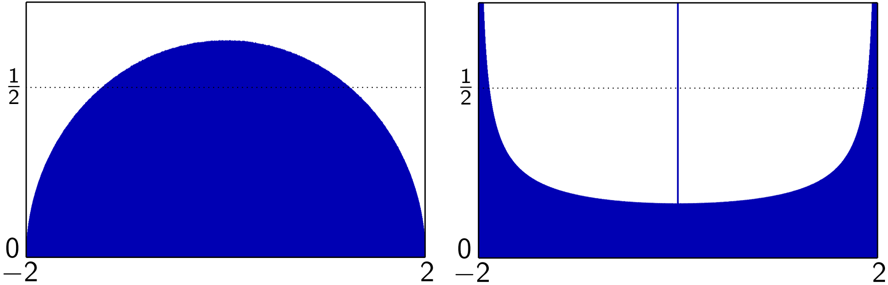

Figure: Distribution of $\frac{\#X(\mathbb{F}_p)-p-1}{\sqrt{p}}$ for $p \to \infty$ for an ordinary elliptic curve (left) and a CM elliptic curve (right)

The spike has area $\frac{1}{2}$ (!!).

Our original motivation – Picard ranks

Fact

Let X be a $K3$ surface over $\mathbb{Q}$ and p a prime of good reduction. Then

$$\operatorname{rk}\operatorname{Pic} X_{\overline{\mathbb{Q}}} \leq \operatorname{rk}\operatorname{Pic} X_{\overline{\mathbb{F}}_p}.$$

Theorem (F. Charles, 2012)

Let X be a $K3$ surface over $\mathbb{Q}$.

1. *If X has* real multiplication *and* $(22 - \operatorname{rk}\operatorname{Pic} X_{\overline{\mathbb{Q}}})/[E : \mathbb{Q}]$ *is odd then, for every prime p of good reduction,*

$$\operatorname{rk}\operatorname{Pic} X_{\overline{\mathbb{Q}}} + [E : \mathbb{Q}] \leq \operatorname{rk}\operatorname{Pic} X_{\overline{\mathbb{F}}_p}$$

2. *Otherwise, there exists a prime p of good reduction such that*

$$\operatorname{rk}\operatorname{Pic} X_{\overline{\mathbb{F}}_p} = \operatorname{rk}\operatorname{Pic} X_{\overline{\mathbb{Q}}} \quad \text{or} \quad \operatorname{rk}\operatorname{Pic} X_{\overline{\mathbb{F}}_p} = \operatorname{rk}\operatorname{Pic} X_{\overline{\mathbb{Q}}} + 1\,.$$

Hodge structures

Definition (P. Deligne, 1971)

1. A (pure $\mathbb{Q}$-)*Hodge structure* of weight i is a finite dimensional $\mathbb{Q}$-vector space V, together with a decomposition

$$V_\mathbb{C} := V \otimes_\mathbb{Q} \mathbb{C} = H^{0,i} \oplus H^{1,i-1} \oplus \ldots \oplus H^{i-1,1} \oplus H^{i,0}$$

   such that $\overline{H^{m,n}} = H^{n,m}$ for every $m, n \in \mathbb{Z}_{\geq 0}$, $m + n = i$.

2. A *polarisation* on a pure $\mathbb{Q}$-Hodge structure V of even weight is a nondegenerate symmetric bilinear form $\langle .,. \rangle \colon V \times V \to \mathbb{Q}$ such that its $\mathbb{C}$-bilinear extension $\langle .,. \rangle \colon V_\mathbb{C} \times V_\mathbb{C} \to \mathbb{C}$ satisfies the two conditions below.
   - One has $\langle x, y \rangle = 0$ for all $x \in H^{m,n}$ and $y \in H^{m',n'}$ such that $m \neq n'$.
   - The inequality $i^{m-n}\langle x, \overline{x} \rangle > 0$ is true for every $0 \neq x \in H^{m,n}$.

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Facts

1. *Hodge structures of weight i form an abelian category.*
2. *Every polarisable Hodge structure is a direct sum of primitive ones.*

Hodge structures II

Definition (Yu. Zarhin, 1983)

A *Hodge structure of $K3$ type* is a primitive polarisable Hodge structure of weight 2 such that $\dim_{\mathbb{C}} H^{2,0} = 1$.

Examples

Let X be a compact complex manifold that is Kähler.

1. Then $H^j(X, \mathbb{Q})$ is naturally a polarisable pure $\mathbb{Q}$-Hodge structure of weight j. [The polarisation is induced by the cup product.]
2. For X a $K3$ surface, the *transcendental part*

$$T := (\operatorname{Pic} X \otimes_{\mathbb{Z}} \mathbb{Q})^{\perp} \subset H^2(X, \mathbb{Q})$$

is a Hodge structure of $K3$ type.

Real and complex multiplication

Theorem (Yu. Zarhin, 1983)

Let T be a Hodge structure of $K3$ type.

1. *Then $E := \mathrm{End}(T)$ is either a totally real field or a CM field.*
2. *Thereby, every $\varphi \in E$ operates as a self-adjoint mapping. I.e.,*

$$\langle \varphi(x), y \rangle = \langle x, \overline{\varphi}(y) \rangle \,,$$

for $\bar{\ }$ the identity map in the case that E is totally real and the complex conjugation in the case that it is a CM field.

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Definition

If $E \supsetneq \mathbb{Q}$ then one speaks of *real multiplication* when E is totally real and of *complex multiplication* when E is CM.
[The same terminology, applied to a $K3$ surface, means that the associated Hodge structure T has real or complex multiplication.]

RM on $K3$ surfaces

The difference (Recall)

- For X an elliptic curve, one considers $\operatorname{End}(H)$, for $H := H^1(X(\mathbb{C}), \mathbb{Q})$.
- For X a $K3$ surface, consider $\operatorname{End}(T)$, for $T := P^\perp$ the transcendental part of $H^2(X(\mathbb{C}), \mathbb{Q})$.

Questions

- Can one construct $K3$ surfaces having real multiplication?
- How many $K3$ surfaces have real multiplication? I.e., what is the dimension of the corresponding locus in moduli space?
- Are there $K3$ surfaces defined over $\mathbb{Q}$ that have real multiplication?

$K3$ surfaces having CM due to an automorphism

Examples

$$X\colon w^2 = f_6(x,y,z)$$

- with $f_6(x,y,-z) = -f_6(x,y,z)$ or $f_6(y,x,z) = -f_6(x,y,z)$. Automorphism
$$I\colon (w, x\!:\!y\!:\!z) \mapsto (\zeta_4 w, x\!:\!y\!:\!-z) \text{ or}$$
$$I\colon (w, x\!:\!y\!:\!z) \mapsto (\zeta_4 w, y\!:\!x\!:\!z)\,.$$
CM with $\mathbb{Q}(\sqrt{-1})$.
- with $f_6(x,y,z) = \zeta_3 f_6(y,z,x)$. Automorphism
$$I\colon (w, x\!:\!y\!:\!z) \mapsto (\zeta_6 w, y\!:\!z\!:\!x)\,.$$
CM with $\mathbb{Q}(\sqrt{-3})$.

[There are similar examples of degrees 4 and 6.]

The Lefschetz trace formula shows that $I \circ I$ [resp. $I \circ I \circ I$] acts on the transcendental part $T \subset H^2(X(\mathbb{C}), \mathbb{Q})$ non-trivially, with the only eigenvalue (-1). It does so on the [dim 21] orthogonal complement of the inverse image of a general line on $\mathbf{P}^2$.

The period space

Definition

By a *marked $K3$ surface,* we mean a complex $K3$ surface together with an isomorphism $i\colon \mathbb{Z}^{22} \longrightarrow H^2(X, \mathbb{Z})$.

Notation

- The *marking* $i\colon \mathbb{Z}^{22} \to H^2(X, \mathbb{Z})$ determines $c^k := i(e_k) \in H^2(X, \mathbb{Z})$, for $k = 1, \ldots, 22$, which form a basis of $H^2(X, \mathbb{Z})$.
- The dual basis $(c_1, \ldots, c_{22})$ of $H^2(X, \mathbb{Z})$, is given by $(c_k, c^j) = \delta_{kj}$, for $(.\,,.)\colon H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to \mathbb{Z}$ the cup product pairing.
- Moreover, as the pull-back of the cup product pairing via i, the marking distinguishes a perfect, symmetric pairing on $\mathbb{Z}^{22}$.

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- Moreover, as the pull-back of the cup product pairing via i, the marking distinguishes a perfect, symmetric pairing on $\mathbb{Z}^{22}$.

Given any $c \in H^2(X, \mathbb{C})$, there is the *canonical decomposition*

$$c = (c_1, c)c^1 + \cdots + (c_{22}, c)c^{22}\,,$$

of c *with respect to* i.

The period space II

Definition

A marked $K3$ surface (X, i) gives rise to a point

$$\tau(X, i) := ((c_1, [\omega]) : \cdots : (c_{22}, [\omega])) \in \mathbf{P}^{21}(\mathbb{C}),$$

called the *period point* of (X, i).

Here, $[\omega] \in H^2(X, \mathbb{C})$ is the nowhere vanishing holomorphic $(2, 0)$-form, uniquely determined up to scaling.

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Theorem (I.R. Shafarevich, ≈1965)

Let κ be a perfect pairing on $\mathbb{Z}^{22}$.

1. *Let (X_1, i_1) and (X_2, i_2) be marked $K3$ surfaces inducing the pairing κ on $\mathbb{Z}^{22}$ and having the same period point. Then (X_1, i_1) and (X_2, i_2) are isomorphic.*
2. *The set of the period points of all marked $K3$ surfaces inducing the pairing κ is $\Omega_\kappa := \left\{(x_1 : \cdots : x_{22}) \in \mathbf{P}^{21}(\mathbb{C}) \mid \kappa(x, x) = 0, \kappa(x, \overline{x}) > 0\right\}$. This is an open subset of a quadric in $\mathbf{P}^{21}(\mathbb{C})$.*

The period space III

Fact (The relative situation)

Let $q\colon (\mathfrak{X}, \mathfrak{i}) \to Y$ *be a family of marked K3 surfaces. Then the period mapping* $\tau\colon Y \to \mathbf{P}^{21}(\mathbb{C}), t \mapsto \tau(\mathfrak{X}_t, i_t)$, *is holomorphic.*

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Fact (Restricted period space)

Let $r \in \{1, \dots, 20\}$ be an integer and κ a perfect pairing on $\mathbb{Z}^{22}$. Then the set of the period points of all marked K3 surfaces (X, i) such that

- *the classes $c_{22-r+1}, \dots, c_{22} \in H^2(X, \mathbb{Z})$ from the dual basis are algebraic, i.e. in $\operatorname{Pic} X \subset H^2(X, \mathbb{Z})$, and*
- *via i, the pairing κ gets induced,*

is

$$\Omega_{\kappa,r} := \{(x_1 : \dots : x_{22-r} : 0 : \dots : 0) \in \mathbf{P}^{21-r}(\mathbb{C}) \mid \kappa(x,x) = 0, \kappa(x, \overline{x}) > 0\} .$$

This is an open subset of a quadric $Q_{\kappa,r} \subset \mathbf{P}^{21-r}(\mathbb{C})$.

Periods and RM

Theorem

Let $r \in \{1, \ldots, 20\}$, κ be a perfect pairing on $\mathbb{Z}^{22}$, and K be a totally real number field of degree d.

Then there is an at most countable union $M \subseteq Q_{\kappa,r}$ of quadrics of dimensions $\frac{22-r}{d} - 2$ such that the following is true.

- *Let $x \in \Omega_{\kappa,r} \subset Q_{\kappa,r}$ be the period point of a marked K3 surface (X, i), for which $c_{22-r+1}, \ldots, c_{22} \in \operatorname{Pic} X$ and the Picard rank is exactly r.*

  Then X has real multiplication by K if and only if $x \in M$.

Idea of Proof. $T = (\operatorname{Pic} X \otimes_{\mathbb{Z}} \mathbb{Q})^{\perp} = \operatorname{span}(c^1, \ldots, c^{22-r})$. RM by K means that K operates $\mathbb{Q}$-linearly on T, keeping $x = x_1 c^1 + \cdots + x_{22-r} c^{22-r}$ as a simultaneous eigenvector.

There are countably many such operations. Each time, the eigenspaces are of [projective] dimension $\frac{22-r}{d} - 1$. It can be shown that they are not contained in $Q_{\kappa,r}$.

A particular family

Consider double covers of $\mathbf{P}^2_{\mathbb{C}}$, ramified over a union of six lines,

$$X'\colon w^2 = l_1 \cdots l_6\,,$$

for $l_1, \ldots, l_6$ linear forms in three variables.

Assume that no point is contained in three lines. Then there are 15 singular points of type A_1. The minimal desingularisation X is a $K3$ surface.

Proposition (The global holomorphic $(2,0)$-form)

Let X be a $K3$ surface, obtained as the minimal desingularisation of $X'\colon w^2 = l_1 \cdots l_6$, for $l_1, \ldots, l_6$ linear forms in the variables x, y, and z. Assume that no three of the six linear forms have a zero in common.

Then, for any linear form $l \neq x, y$ that defines an irreducible curve on X',

$$\omega' := \frac{d(\frac{x}{l}) \wedge d(\frac{y}{l})}{\frac{w}{l^3}}$$

is a differential form on X', the pull-back ω of which to X is a global holomorphic $(2,0)$-form without zeroes or poles.

Spheroids representing transcendental classes

The surfaces considered generically have Picard rank 16.

There are 16 obvious algebraic classes $L, E_1, \ldots, E_{15}$, generating a rank-16 submodule $P \subset H^2(X, \mathbb{Z})$. **We explicitly need six further generators.**

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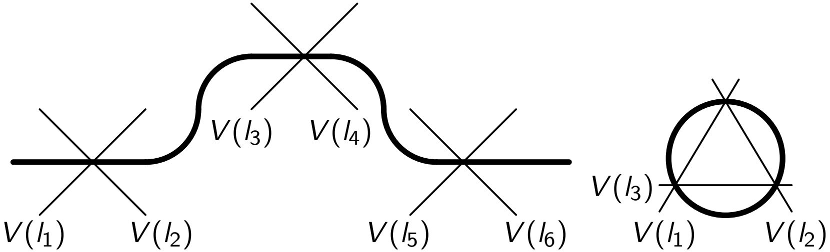

Figure: A deformed line and a curve encircling a triangle

Assume that the branch locus is the union of six *real* lines [no three of which have a point in common].

We start with 1-manifolds in $\mathbf{P}^2(\mathbb{R})$ as in the pictures above. I.e. such meeting the branch locus $V(l_1 \cdots l_6)$ only in its double points. We also allow 1-manifolds that encircle a quadrangle or pentagon instead of a triangle.

Spheroids representing transcendental classes II

Parametrised by a differentiable map $\gamma\colon \mathbb{R}/{\sim} \to \mathbf{P}^2(\mathbb{R})$, for $\sim$ the equivalence relation, given by $t \sim t' \Leftrightarrow t - t' \in \mathbb{Z}$.

On a suitable affine chart of $\mathbf{P}^2(\mathbb{R})$, one has a map $\underline{\gamma}\colon \mathbb{R}/{\sim} \to \mathbb{R}^2$.

Extend $\underline{\gamma}$ to a differentiable map in two variables by putting

$$\gamma'\colon \mathbb{R}/{\sim} \times \mathbb{R} \longrightarrow \mathbb{C}^2 \subset \mathbf{P}^2(\mathbb{C}), \qquad (t, u) \mapsto \gamma_0(t) + \mathrm{i}ub\,, \tag{1}$$

for a suitable vector $b \in \mathbb{R}^2$.

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for a suitable vector $b \in \mathbb{R}^2$.

Then $\lim\limits_{u\to\pm\infty} \gamma'(t,u)$ exists in $\mathbf{P}^2(\mathbb{C})$ and is independent of t.
Therefore, γ' actually provides a continuous map α' from

$$\mathbb{R}/{\sim} \times [-\infty,\infty]/(\mathbb{R}/{\sim} \times \{\infty\}\,, \mathbb{R}/{\sim} \times \{-\infty\}) \cong S(\mathbb{R}/{\sim}) = S(S^1) = S^2 \tag{2}$$

to $\mathbf{P}^2(\mathbb{C})$. [α' is differentiable outside the two poles $n, s \in S^2$.]

Spheroids representing transcendental classes III

Proposition (Lifting to the double cover)

Let $V(l_1), \ldots, V(l_6)$ be six real lines in $\mathbf{P}^2$ such that no three of them have a point in common. Take an affine chart that meets each of the six lines.

1. *Then there is a union $E \subset \mathbb{R}^2$ of finitely many 1-dimensional subvector spaces such that, for $b \in \mathbb{R}^2 \setminus E$, each of the spheroids α', as constructed above, meets $V(l_1 \cdots l_6)$ only in the three to five real points.*
2. *Assume that $b \in \mathbb{R}^2 \setminus E$. Then the continuous map $\alpha' \colon S^2 \to \mathbf{P}^2(\mathbb{C})$, as constructed in (1) and (2), lifts to a continuous map $\widetilde{\alpha} \colon S^2 \to X'$, for $X' \colon w^2 = l_1 \cdots l_6$ the double cover.*

Idea of Proof. 1. Direct calculation.

2. Let $x_1, \ldots, x_m \in S^2$, $m = 3, 4, 5$, be the points mapped to the ramification locus. Then essentially $\alpha_0 := \alpha'|_{S^2 \setminus \{x_1, \ldots, x_m\}}$ needs to be lifted.

For this, one only has to verify

$$(\alpha_0)_\# \pi_1(S^2 \setminus \{x_1, \ldots, x_n\}, .) \subseteq \pi_\# \pi_1(X_0, .)$$

for $X_0 := \pi^{-1}(\mathbf{P}^2(\mathbb{C}) \setminus V(l_1 \cdots l_6))$, which is completely local.

Spheroids representing transcendental classes IV

- The spheroid $\widetilde{\alpha}\colon S^2 \to X'$ represents a class in $\pi_2(X', .)$.
- X' and the $K3$ surface X are simply connected. By Hurewicz's Theorem,
$$\pi_2(X', .) \cong H_2(X', \mathbb{Z}) \quad \text{and} \quad \pi_2(X, .) \cong H_2(X, \mathbb{Z})\,.$$
- As $H_2(X', \mathbb{Z}) = H_2(X, \mathbb{Z})/[E_1, \ldots, E_{15}]$, $\widetilde{\alpha}$ can [non-uniquely] be lifted to X.

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Notation (Explicit cohomology classes)

A spheroid α defines a cohomology class

$$c_\alpha := \alpha_!(1) \in H^2(X, \mathbb{Z}) \, ,$$

for $1 \in H^0(S, \mathbb{Z})$ the canonical generator.

The construction described provides by far more than six representatives of classes in $H^2(X, \mathbb{Z})/P$.

The cup product

Fact

Let $c_\alpha \in H^2(X, \mathbb{Z})$ be given by a spheroid and $w \in H^2(X, \mathbb{C})$ be represented by the smooth 2-form ω. Then, for the cup product pairing, one has

$$(c_\alpha, w) = \int_{S^2} \alpha^* \omega \,.$$

Idea of Proof. $(c_\alpha, w) = \langle w \cup c_\alpha, [X] \rangle = \langle w \cup \alpha_!(1), [X] \rangle = \langle \alpha^* w, [S^2] \rangle$. Here, the final equality is

$$(w \cup \alpha_!(1)) \cap [X] = w \cap (\alpha_!(1) \cap [X]) = w \cap \alpha_*([S^2]) = \alpha_*(\alpha^* w \cap [S^2]) \,.$$

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□

Remark

In order to numerically calculate these 2-dimensional integrals, we use Fubini's Theorem in a naive manner. The resulting 1-dimensional integrals are evaluated using the Gauß-Legendre method of a degree between 30 and 300.

The cup product II

Algorithm (Determining the cup product on $P^\perp$, up to scaling)

Let X be a $K3$ surface that is given as the minimal desingularisation of a double cover of the form $w^2 = xyz(x+y+z)(x+a_0y+b_0z)(x+c_0y+d_0z)$ and $\alpha_1, \dots, \alpha_n \colon S^2 \to X$ be spheroids.

1. Choose open neighbourhoods $\mathbb{D} \cong U(a_0) \ni a_0, \dots, \mathbb{D} \cong U(d_0) \ni d_0$ in such a way that, for every $(a, b, c, d) \in \mathbb{D}^4$, no three of the resulting six lines in $\mathbf{P}^2_{\mathbb{C}}$ have a point in common. Then the $c_{\alpha_1}, \dots, c_{\alpha_n}$ uniquely extend to the whole family of surfaces $X_{(a,b,c,d)}$ analogous to X.

   Moreover, choose N surfaces $X_1, \dots, X_N$ at random from the family and write down the corresponding holomorphic 2-forms $\omega_1, \dots, \omega_N$. [We work with $N = 50$.]

2. Set up the matrix $M := (\langle c_{\alpha_j}, \omega_i \rangle)_{1 \leq i \leq N, 1 \leq j \leq n}$ using numerical integration and calculate the singular value decomposition of M. Six singular values should be numerically nonzero. The others give rise to linear relations among the cohomology classes $c_{\alpha_1}, \dots, c_{\alpha_n}$.

The cup product III

3. Choose a basis $c_1, \ldots, c_6$ of the free $\mathbb{Z}$-module spanned by $c_{\alpha_1}, \ldots, c_{\alpha_n}$ modulo the relations found.
4. Build from M the matrix $F := (\langle c_{j_1}, \omega_i \rangle \langle c_{j_2}, \omega_i \rangle)_{1 \leq i \leq N, 1 \leq j_1 \leq j_2 \leq 6}$ and determine an approximate solution of the corresponding homogeneous linear system of equations, using the QR-factorisation of F. The solution vector describes the symmetric, bilinear form desired.

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Remarks

- As it relies on the restricted period space, our method detects only the restriction of the cup product pairing to $P^\perp \subset H^2(X, \mathbb{Q})$. Every class in $H^2(X, \mathbb{Z})/P$ has a unique representative in $P^\perp$, the orthogonal projection $\mathrm{pr}\colon H^2(X, \mathbb{Z})/P \to P^\perp$ being injective.
- Let $\alpha\colon S^2 \to X$ be constructed from a deformed line, as above. Then $(\mathrm{pr}(c_\alpha), \mathrm{pr}(c_\alpha)) = -\frac{1}{2}$. [We use this observation for scaling.]

The cup product IV

Theorem (Proof depending on numerical integration)

Let X be the minimal desingularisation of a double cover of $\mathbf{P}^2_{\mathbb{C}}$ ramified over a union of six real lines, such that no three of them have a point in common. Then the classes of the spheroids, as described above, always generate the whole of $H^2(X, \mathbb{Z})/P$.

Tracing the preimage of a curve in the period space

In a family of $K3$ surfaces of Picard rank 16, there are **1-dimensional** subfamilies having real multiplication by $\mathbb{Q}(\sqrt{d})$

Strategy

Let X be an isolated example of a $K3$ surface of type

$$X_{(a_0,b_0,c_0,d_0)}\colon w^2 = xyz(x+y+z)(x+a_0y+b_0z)(x+c_0y+d_0z)$$

that has real multiplication by a quadratic field $\mathbb{Q}(\sqrt{d})$. The strategy below describes how to find the 1-dimensional family of RM surfaces, X belongs to.

1. Run the Algorithm above to fix a marking i on X and to calculate the cup product pairing in terms of i.

   Then $\mathbb{D} \cong U(a_0) \ni a_0, \ldots, \mathbb{D} \cong U(d_0) \ni d_0$ are chosen in such a way that, for every $(a,b,c,d) \in \mathbb{D}^4$, no three of the resulting six lines in $\mathbf{P}^2_{\mathbb{C}}$ have a point in common. Thus, the marking extends to the whole family and there is the associated period map

$$\tau\colon \mathbb{D}^4 \longrightarrow Q, \quad (a,b,c,d) \mapsto \tau(X_{(a,b,c,d)}, i_{(a,b,c,d)})\,.$$

Tracing the preimage of a curve in the period space II

2. Calculate the period point of $X = X_{(a_0,b_0,c_0,d_0)}$ and identify the three linear relations between the six periods that encode real multiplication. These define, together with the cup product pairing, a conic C in the restricted period space.
3. Trace the curve $Q^{-1}(C) \subset \mathbb{D}^4$ using a numerical continuation method.
4. Use the singular-value decomposition in order to find algebraic relations between the coordinates of the points found. Control, using Gröbner bases, that they indeed define an algebraic curve.

A result

Theorem (E.+J., 2017)

Consider the family of double covers $X'_{(a,b,c,d)}$ *of* $\mathbf{P}^2$, *given by*

$$w^2 = (x + ay + bz)(x + cy + dz)f_4(x, y, z)\,,$$

for $f_4 := x^4 - 2x^3y - 5x^2y^2 - 26x^2z^2 + 6xy^3 + 104xyz^2 + 9y^4 - 130y^2z^2 + 52z^4$.

1. *Then the branch locus is the union of six lines, which are in general position for a generic choice of* $(a, b, c, d) \in \mathbb{C}^4$. *In this case, the minimal desingularisation* $X_{(a,b,c,d)}$ *of* $X'_{(a,b,c,d)}$ *is a* $K3$ *surface of Picard rank* 16.
2. *Consider the closed subscheme* $C \subset \mathbf{A}^4$, *given by the equations*

$$\begin{aligned}
0 &= 630\,272a - 11\,421bd^5 + 411\,400bd^3 - 871\,552bd - 272\,976c^2d^2 + 315\,136c^2 \\
&\qquad + 98\,982cd^4 - 3\,508\,064cd^2 + 2\,205\,952c + 233\,496d^4 - 6\,409\,856d^2 + 4\,411\,904\,, \\
0 &= 78\,784bc - 243bd^4 + 37\,040bd^2 + 110\,528b - 5808c^2d + 2106cd^3 - 319\,792cd + 4968d^3 - 714\,688d\,, \\
0 &= 243bd^6 - 8960bd^4 + 29\,952bd^2 - 26\,624b + 5808c^2d^3 - 11\,648c^2d - 2106cd^5 \\
&\qquad + 76\,432cd^3 - 144\,768cd - 4968d^5 + 140\,608d^3 - 259\,584d\,, \\
0 &= 2c^3 + 28c^2 - 3cd^2 + 98c - 8d^2 + 104\,.
\end{aligned}$$

Then C *is a geometrically irreducible, nonsingular curve of genus* 1.

A result II

3. There is strong evidence that, for generic $(a, b, c, d) \in C(\mathbb{C})$, the $K3$ surface $X_{(a,b,c,d)}$ is of Picard rank 16 and has real multiplication by $\mathbb{Q}(\sqrt{13})$.

Proof. 1) For $(a, b, c, d) := (0, -4, -9, -8)$, an isolated example appears, which we had found before by a different approach.

2) This is easily obtained by a calculation in any computer algebra system. We used `magma` for this purpose.

3) The curve C is the result of the Strategy above, taking the isolated example above as the starting point. Evidence for assertion c) includes that one has $\#X_\xi(\mathbb{F}_p) \equiv 1 \pmod p$ for every prime number $p < 500$ such that $p \equiv 2, 5, 6, 7, 8, 11 \pmod{13}$ and every point $\xi \in C(\mathbb{F}_p)$. □

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Remark

The genus 1 curve C has $\mathbb{Q}$-rational points. Taking any of them as the origin, the Mordell–Weil group of C is isomorphic to $\mathbb{Z}$.

A result III

Remarks (Some details)

- [Transcendental classes.] We worked with 14 classes, which were represented by spheroids, as explained above.

  In step 2 of the Algorithm, we found six singular values within a factor of 100, while the next one was smaller by nine orders of magnitude. In the basis chosen, the cup product form found on $P^\perp$ has only coefficients from $\{\pm 1, \pm \frac{1}{2}, 0\}$, up to errors that are smaller than 10^{-10}.

- [Tracing the curve.] In an expert's language, we applied a predictor–corrector method. More precisely, we used the Euler predictor, followed by Newton corrector steps.

  For numerical integration, the Gauß-Legendre method of degree 100 [i.e. order 200] was used. Based on this, we determined 101 points on $Q^{-1}(C) \subset \mathbb{D}^4$, each with a numerical precision of 80 digits.

A result IV

- Polynomials of degree ≤ 3 in four variables form a vector space of dimension 35. When looking for cubic relations between the 101 points found, we ended up with 25 singular values in the range from 1714 to $6.08 \cdot 10^{-41}$, the other ten being less than 10^{-80}.

  Thus, the curve sought is contained in an intersection of ten cubics in $\mathbf{A}^4$. The equations given form a Gröbner basis for the ideal generated by them.

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Thank you!!!