On integral points on open degree four del Pezzo surfaces

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joint work with
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Diophantine equations

Problem (Diophantine equation)

Given $f \in \mathbb{Z}[X_1, \ldots, X_n]$, describe the set

$$L(f) := \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid f(x_1, \ldots, x_n) = 0\}$$

explicitly.

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Geometric Interpretation

Integral points on a hypersurface in A^n.

Seemingly easier problem: Decide whether $L(f)$ is non-empty.

Statistical heuristics

Given a concrete f, how many solutions do we *naively* expect?

Put $Q(B) := \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid |x_i| \leqslant B\}$. Then

$$\#Q(B) = (2B+1)^n \sim C_1 \cdot B^n .$$

On the other hand,

$$\max_{(x_1,\ldots,x_n)\in Q(B)} |f(x_1, \ldots, x_n)| \sim C_2 \cdot B^{\deg f} .$$

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Heuristics

Assuming equidistribution of the values of f on $Q(B)$, we are therefore led to expect the asymptotics

$$\#\{(x_1, \ldots, x_n) \in V_f(\mathbb{Z}) \mid |x_1|, \ldots, |x_n| \leqslant B\} \sim C \cdot B^{n - \deg f}$$

for the number of solutions.

Statistical heuristics–Examples

The statistical heuristics explains the following well-known examples.

Examples

- $n - \deg f < 0$: *Log general type,*
  Very few solutions.

  Example: $X_2^2 - 2X_1^3 = 1$.
  Integral points on an elliptic curve (Siegel).
- $n - \deg f = 0$: *Log intermediate type,*
  A few solutions.

  Examples: $X_2^2 - 2X_1^2 = 1$, $X_1^3 + X_2^3 + X_3^3 = 3$.
  Pell equations. Integral points on conics (Gauß). Three cubes problem.
- $n - \deg f > 0$: *Log Fano varieties,*
  Many solutions.

  Example: $X_1^2 + X_2^2 = X_3^2$ or $X_1^2 + X_2^2 - 10X_3^2 = 3$.
  Representation of an integer by a ternary quadratic form.

Statistical heuristics–Refinement

We are mainly interested in varieties of log intermediate type.

Heuristics (Refinement for varieties of log intermediate type)

Assume that the projective closure $\widetilde{V}_f \supset V_f$, $\widetilde{V}_f \subset \mathrm{P}^n_{\mathbb{Q}}$ is non-singular and that $\operatorname{rk}\operatorname{Pic}\widetilde{V}_f = r$. Then one is led to expect the asymptotics

$$\#\{(x_1, \ldots, x_n) \in V_f(\mathbb{Z}) \mid |x_1|, \ldots, |x_n| \leqslant B\} \sim C \cdot (\log B)^{r-1}$$

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Assume that the projective closure $\widetilde{V}_f \supset V_f$, $\widetilde{V}_f \subset \mathrm{P}^n_{\mathbb{Q}}$ is non-singular and that $\operatorname{rk}\operatorname{Pic}\widetilde{V}_f = r$. Then one is led to expect the asymptotics

$$\#\{(x_1,\ldots,x_n) \in V_f(\mathbb{Z}) \mid |x_1|,\ldots,|x_n| \leqslant B\} \sim C\cdot(\log B)^{r-1}$$

for the number of solutions.

Indeed, Manin's conjecture predicts $C\cdot B(\log B)^{r-1}$ rational points

$$(x_0 : x_1 : \ldots : x_n) \in \widetilde{V}_f(\mathbb{Z})$$

of height $\leqslant B$ and, among them, exactly those with $x_0 = \pm 1$ are integral.

Complications

Despite these heuristics, it might happen that there are *no* integral points, for several reasons.

Three kinds of reasons are known from the situation of rational points.

- *p*-adic insolubility,
  $2X_1^3 + 7X_2^3 + 14X_3^3 + 49X_4^3 + 98X_5^3 = 1.$
- Insolubility in reals,
  $X_1^2 + X_2^2 = -1.$
- Brauer-Manin obstruction

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Concerning integral points, (in)solubility in reals is a greater issue than for rational points.

Strong obstruction at infinity

Examples

1. $U_1 \subset \mathrm{A}^2_{\mathbb{Z}} \colon X_1^2 + X_2^2 = 65,$
2. $U_2 \subset \mathrm{A}^3_{\mathbb{Z}} \colon 2X_1^2 + X_2^2 + X_3^2 = 26,$
   $3X_2^2 + X_3^2 + X_4^2 = 13.$

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Both varieties are *strongly obstructed* at infinity. I.e., the real manifolds $U_1(\mathbb{R}) \subset \mathbb{R}^2$ and $U_2(\mathbb{R}) \subset \mathbb{R}^3$ are both bounded.

For integral points, this leaves us with only finitely many cases, $U_1(\mathbb{Z}) = \{(\pm 1, \pm 8), (\pm 4, \pm 7), (\pm 7, \pm 4), (\pm 8, \pm 1)\}$, $U_2(\mathbb{Z}) = \varnothing$.

U_2 has $\mathbb{Q}$-rational points and $\mathbb{Z}_p$-valued points for every prime number p. E.g., $(\frac{18}{7}, \frac{1}{7}, \frac{25}{7}, \frac{3}{7})$ and $(\frac{54}{19}, \frac{23}{19}, \frac{55}{19}, \frac{9}{19})$.

U_2 is an *open del Pezzo surface* of degree 4.

Weak obstruction at infinity

Examples

1. $U_1\colon X_1^2 - X_2^2 = 3,$
2. $U_2\colon ((11X_1 + 5)X_2 + 3)X_3 = 3X_1 + 1.$ (Y. Harpaz, 2015)

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$U_1(\mathbb{Z}) = \{(\pm 2, \pm 1)\}$.

$U_2(\mathbb{Z}) = \varnothing$: Every real point $x = (x_1, x_2, x_3) \in U(\mathbb{R})$ must fulfil

$$x_1\left(11 - \frac{3}{x_2x_3}\right) = \frac{1}{x_2x_3} - \frac{3}{x_2} - 5\,.$$

This immediately shows that $|x_2|, |x_3| \geqslant 1$ implies $|x_1| \leqslant \frac{9}{8}$.

$x_1 = 0, \pm 1$ does not yield any solutions.

Both examples are *weakly obstructed* at infinity. I.e., contained in a union of finitely many tubular neighbourhoods of algebraic hypersurfaces, the hypersurfaces themselves not enclosing U,

$$U_j(\mathbb{R}) \subseteq \bigcup_{i=1}^{N} \{x \in \mathrm{A}^n(\mathbb{R}) \mid |P_i(x)| \leqslant c_i\}\,.$$

Weak obstruction at infinity II

Theorem (J. + D. Schindler, 2015)

U being weakly obstructed at infinity implies (for $U(\mathbb{R})$ connected) that $U(\mathbb{Z})$ is not Zariski-dense in U.

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Theorem (J. + D. Schindler, 2015)

Let $X \subset \mathrm{P}^n_{\mathbb{Q}}$ be a normal, projective variety, $l \in \Gamma(\mathrm{P}^n, \mathscr{O}(1))$ a linear form, $H := V(l) \subset \mathrm{P}^n$ the corresponding hyperplane, and put $U := X \backslash H$.

Suppose that

- *the scheme $(H \cap X)_{\mathbb{R}}$ is reduced and irreducible and that*
- *every connected component of $U(\mathbb{R})$ has a limit point $x \in (H \cap X)(\mathbb{R})$ that is non-singular as a point on $H \cap X$.*

Then U is not (weakly) obstructed at ν.

Remark

Y. Harpaz' example is a normal cubic surface, but $H \cap X$ is a union of three lines. Thus, the theorem does not apply.

Brauer-Manin obstruction

Let U be a scheme of finite type over a number field k and

$$\alpha \in \mathrm{Br}(U) = H^2_{\text{ét}}(U, \mathbb{G}_m)$$

a Brauer class.

At each place ν of k, one has a *local evaluation map*

$$\begin{aligned} \mathrm{ev}_{\alpha,\nu} \colon U(k_\nu) &\longrightarrow \mathbb{Q}/\mathbb{Z}\,, \\ x &\mapsto \alpha|_x\,. \end{aligned}$$

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Facts (Yu. I. Manin, ≈1970)

- *The local evaluation map is locally constant with respect to the ν-adic topology.*
- *If U is proper then $\operatorname{ev}_{\alpha,\nu}$ is constantly zero for almost all places ν.*

Thus, an adelic point $(x_\nu)_\nu \in U(\mathbb{A}_k)$ such that $\sum_\nu \operatorname{ev}_{\alpha,\nu}(x_\nu) \neq 0$ cannot be approximated by rational points. This is called the Brauer-Manin obstruction.

Brauer-Manin obstruction to integral points

Fact (J.-L. Colliot-Thélène and F. Xu, 2009)

Choose a model *of* U*, an* $\mathscr{O}_K$*-scheme* $\mathscr{U}$ *of finite type the generic fibre of which is* U*.*

Then the local evaluation map

$$\mathrm{ev}_{\alpha,\nu}\colon \mathscr{U}(\mathscr{O}_k) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

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is constantly zero for almost all places ν.

Thus, there is a Brauer-Manin obstruction to integral points,

- to strong approximation,
- to the integral Hasse principle.

The Brauer group

The Hochschild-Serre spectral sequence

$$H^p(\mathrm{Gal}(\overline{k}/k), H^q_{\text{ét}}(U_{\overline{k}}, \mathbb{G}_m)) \Longrightarrow H^{p+q}_{\text{ét}}(U, \mathbb{G}_m)$$

yields a three-step filtration

$$0 \subseteq \mathrm{Br}_0(U) \subseteq \mathrm{Br}_1(U) \subseteq \mathrm{Br}(U)\,.$$

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Assumption

$$\Gamma_{\text{ét}}(U_{\overline{k}}, \mathbb{G}_m) = \overline{k}^*\,. \tag{1}$$

- $\mathrm{Br}_0(U)$ is the image of a natural homomorphism

$$H^2(\mathrm{Gal}(\overline{k}/k), \Gamma_{\text{ét}}(U_{\overline{k}}, \mathbb{G}_m)) = \mathrm{Br}(k) \longrightarrow \mathrm{Br}(U)\,.$$

This is an injection as soon as U has an adelic point.

$\mathrm{Br}_0(U)$ does not contribute to the Brauer-Manin obstruction.

The Brauer group II

- One has $H^3(\mathrm{Gal}(\overline{k}/k), \Gamma_{\text{ét}}(U_{\overline{k}}, \mathbb{G}_m)) = H^3(\mathrm{Gal}(\overline{k}/k), \overline{k}^*) = 0$ when k is a number field. Thus

$$\mathrm{Br}_1(U)/\,\mathrm{Br}_0(U) \cong H^1(\mathrm{Gal}(\overline{k}/k), \mathrm{Pic}(U_{\overline{k}}))\,.$$

This subquotient is called the *algebraic* (part of the) Brauer group.

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  This subquotient is called the *algebraic* (part of the) Brauer group.

- $\mathrm{Br}_1(U)$ is the kernel of the natural homomorphism $\mathrm{Br}(U) \to \mathrm{Br}(U_{\overline{k}})$. Thus, there is a natural injection

$$\mathrm{Br}(U)/\,\mathrm{Br}_1(U) \hookrightarrow \mathrm{Br}(U_{\overline{k}})^{\mathrm{Gal}(\overline{k}/k)}\,.$$

  This quotient is called the *transcendental* (part of the) Brauer group.

  It seems hard to decide which Galois invariant Brauer classes on $U_{\overline{k}}$ descend to U. Partial results:

  - Colliot-Thélène, J.-L. and Skorobogatov, A. N.: Descente galoisienne sur le groupe de Brauer, *J. Reine Angew. Math.* 682 (2013), 141-165.

Degree four del Pezzo surfaces

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- These are non-singular intersections of two quadrics in P^4.
- Geometrically: P^2 blown up in five points in general position.

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- Geometrically: P^2 blown up in five points in general position.

Contains exactly 16 lines, which generate the Picard group.

The group of permutations respecting the intersection matrix is $W(D_5)$ of order 1920.

The pencil of quadrics in P^4 contains exactly five degenerate ones (rank 4). $W(D_5) \cong (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ permutes them via the surjection to S_5.

Our examples

$U := X \setminus H$ for X a degree four del Pezzo surface and H a hyperplane section. We assume $D := H \cap X$ to be a geometrically irreducible curve.

Then

- $D_{\overline{k}}$ is an irreducible divisor such that $D_{\overline{k}}^2 = 4 \neq 0$, hence non-principal. In particular, Assumption (1) is fulfilled.
- $\operatorname{Pic} U_{\overline{k}} = \operatorname{Pic} X_{\overline{k}} / \langle H \rangle \cong D_5^*$.

Algebraic Brauer classes

Observations

- $W(D_5)$ has exactly 197 conjugacy classes of subgroups.
- $H^1(H, D_5^*)$ is
  - 0 in 59 cases [including $H = W(D_5)$, index two, or the trivial group],
  - $\mathbb{Z}/2\mathbb{Z}$ in 62 cases,
  - $(\mathbb{Z}/2\mathbb{Z})^2$ in 44 cases,
  - $(\mathbb{Z}/2\mathbb{Z})^3$ in 16 cases,
  - $(\mathbb{Z}/2\mathbb{Z})^4$ in three cases,
  - $\mathbb{Z}/4\mathbb{Z}$ in nine cases,
  - $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ in three cases, and
  - $(\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z}$ in one case.

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  - $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ in three cases, and
  - $(\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z}$ in one case.

Remark

The Brauer group of a proper degree four del Pezzo surface may be only 0, $\mathbb{Z}/2\mathbb{Z}$, or $(\mathbb{Z}/2\mathbb{Z})^2$.

Algebraic Brauer classes II

Theorem

Let $X \subset \mathrm{P}^4_k$ *be a degree four del Pezzo surface over a number field* k*,* $H := V(l) \subset \mathrm{P}^4_k$ *a* k*-rational hyperplane such that* $H \cap X$ *is geometrically irreducible, and put* $U := X \backslash H$*. Suppose that*

1. *the Galois group operating on the 16 lines on* X *is the index five subgroup in* $W(D_5)$*. Then* $\mathrm{Br}_1(U)/\mathrm{Br}_0(U) = \mathbb{Z}/2\mathbb{Z}$*.*
2. *two of the five degenerate quadrics in the pencil associated with* X *are defined over* k *and the Galois group operating on the 16 lines on* X *is of index 20 in* $W(D_5)$*. Then* $\mathrm{Br}_1(U)/\mathrm{Br}_0(U) = (\mathbb{Z}/2\mathbb{Z})^2$*.*

Algebraic Brauer classes II

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Let $X \subset \mathrm{P}^4_k$ be a degree four del Pezzo surface over a number field k, $H := V(l) \subset \mathrm{P}^4_k$ a k-rational hyperplane such that $H \cap X$ is geometrically irreducible, and put $U := X \backslash H$. Suppose that

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2. *two of the five degenerate quadrics in the pencil associated with X are defined over k and the Galois group operating on the 16 lines on X is of index 20 in $W(D_5)$. Then $\mathrm{Br}_1(U)/\,\mathrm{Br}_0(U) = (\mathbb{Z}/2\mathbb{Z})^2$.*

Remark (Generators, Colliot-Thélène-Xu, 2009)

1. If the pencil contains the k-rational rank 4 quadric $l_1 l_2 - l_3^2 + d l_4^2$ then the quaternion algebra $(\frac{l_1}{l}, d)$ defines an algebraic 2-torsion Brauer class $\alpha \in \mathrm{Br}(U)$.
2. $\mathrm{ev}_{\alpha,p}$ is constantly zero if $l_1, \ldots, l_4$ are linearly independent modulo p and the cusp defined by $l_1 = \ldots = l_4 = 0$ does not lie on U_p.

Transcendental Brauer classes

Lemma

Let k be a number field and $U := X \backslash H$, for $X \subset \mathrm{P}^4_k$ a degree four del Pezzo surface and $H \subset \mathrm{P}^4_k$ a k-rational hyperplane such that $D := H \cap X$ is non-singular. Then there is a canonical monomorphism

$$\mathrm{Br}(U)/\,\mathrm{Br}_1(U) \hookrightarrow J(D)(k)_{\mathrm{tors}}\,,$$

for $J(D)$ the Jacobian variety of D.

Transcendental Brauer classes II

Theorem (J. + D. Schindler, 2015)

Let k be any field and $X \subset \mathrm{P}^4_k$ a del Pezzo surface of degree four over k that is given by a system of equations of the type

$$l_1 l_2 + au^2 = X_0 l_3 \, ,$$
$$l_3 l_4 + bv^2 = X_0 l_1 \, ,$$

for linear forms $l_1, \ldots, l_4, u, v$, and $a, b \in k^*$. Assume that the forms l_1, l_3, u, and v are linearly independent. Put $U := X \backslash H$ for $H := V(X_0) \subset \mathrm{P}^4_k$.

1. *Then the quaternion algebra*
$$\left(\frac{bl_1}{X_0}, \frac{al_3}{X_0}\right)$$
defines a Brauer class $\tau \in \mathrm{Br}(U)_2$.
2. *If $D := V(X_0) \cap X$ is geometrically integral and $\frac{l_1}{l_3}$ is not the square of a rational function on $D_{\overline{k}}$ then τ is transcendental.*

Observe that on the genus one curve $D_{\overline{k}}$, the rational function $\frac{l_1}{l_3}$ has two double zeroes and two double poles, but nevertheless is not a square.

An example

Example

Let the degree four del Pezzo surface $X \subset \mathrm{P}^4_{\mathbb{Q}}$ be given by the system of equations

$$X_1X_4 + X_2^2 = X_0X_3\,,$$
$$X_3(2X_1 + X_2 + X_3) + X_4^2 = X_0X_1.$$

and put $U := X \backslash H$ for $H := V(X_0)$.

1. Then the manifold $X(\mathbb{R})$ is connected, its submanifold $U(\mathbb{R})$ is connected, too, and U is not (weakly) obstructed at ∞.
2. However, strong approximation on U off $\{17, \infty\}$ is violated.

   A $\mathbb{Z}[\frac{1}{17}]$-valued point such that $x_1 \neq 0$ and $x_3 \neq 0$ must necessarily fulfil $(\frac{x_1}{x_0}, \frac{x_3}{x_0})_2 = 1$, although not all $\mathbb{Z}_2$-valued points satisfy this relation.

Idea of proof.

- A Gröbner base calculation shows that the Galois group operating on the 16 lines is the full $W(D_5)$.
- $D := H \cap X$ is a non-singular genus one curve such that $\#D(\mathbb{Q}) = 2$, $D(\mathbb{Q}) = \{(0\!:\!1\!:\!0\!:\!0\!:\!0), (0\!:\!-1\!:\!0\!:\!2\!:\!0)\}$.
- $(\frac{x_1}{x_0}, \frac{x_3}{x_0})$ defines a transcendental Brauer class $\tau \in \operatorname{Br}(U)$.

Therefore, $\operatorname{Br}(U)/\operatorname{Br}(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$, τ being a generator.

The Brauer class τ works only at the prime 2. In particular, the equations do not allow real points such that $\frac{x_1}{x_0}$ and $\frac{x_3}{x_0}$ are both negative.

$\mathrm{ev}_{\tau,17}$ is constant even on rational points. Note that D has bad reduction at 17. □

An example III

Remark

- There are infinitely many integral points on $\mathscr{U}$. The curve $V(X_3)$ yields the family
$$(1:n^4:\pm n^3:0:-n^2)\,.$$
- The curve $V(2X_1+X_2+X_3)$ is elliptic. It carries the six integral points $(1:0:0:0:0)$, $(1:0:-1:1:0)$, $(1:4:-1:-7:-2)$, $(1:4:0:-8:-2)$, $(1:196:-49:-343:-14)$, $(1:196:48:-440:-14)$, and no others.
- A search for integral points on $\mathscr{U}$ delivered 28 of height $<50\,000$ that are not of the forms mentioned above. These are

$(1:-2:1:1:0)$, $(1:-1:-1:2:-1)$, $(1:-3:1:4:-1)$, $(1:-8:6:4:4)$, $(1:4:4:-8:-6)$, $(1:18:-11:-23:-8)$, $(1:-28:20:8:14)$, $(1:-56:30:4:16)$, $(1:-696:-230:4:76)$, $(1:521:-223:-808:-97)$, $(1:1413:381:-3204:-105)$, $(1:-829:467:62:263)$, $(1:912:712:-128:-556)$, $(1:1278:-951:-423:-708)$, $(1:-1595:1157:444:839)$, $(1:-1648:-1288:4352:1004)$, $(1:3573:-2721:-2988:-2073)$, $(1:-6876:3924:9288:2238)$, $(1:3840:-2948:-3056:-2264)$, $(1:5832:4122:-15228:-2916)$, $(1:-15678:-7219:289:3324)$, $(1:-6183:-4899:16344:3879)$, $(1:14688:8947:-791:-5450)$, $(1:11231:-8077:-2950:-5809)$, $(1:-16476:12115:5017:8908)$, $(1:6948:8415:-15687:-10194)$, $(1:-38044:29087:31097:22238)$, and $(1:44152:-34138:-33148:-26396)$.

A modification of the example

Example

Let $\mathscr{X}' \subset \mathrm{P}^4_{\mathbb{Z}}$ be given by the system of equations

$$\begin{aligned} (8X_1 + 3X_0)X_4 + X_2^2 &= X_0(8X_3 + 2X_0)\,, \\ (8X_3 + 2X_0)(16X_1 + X_2 + 8X_3 + 8X_0) + X_4^2 &= X_0(8X_1 + 3X_0)\,. \end{aligned}$$

Put $\mathscr{U}' := \mathscr{X}' \backslash \mathscr{H}$ for the hyperplane $\mathscr{H} := V(X_0) \subset \mathrm{P}^4_{\mathbb{Z}}$ and denote the generic fibre of $\mathscr{U}'$ by U'.

Then $\mathscr{U}'(\mathbb{Z}_p) \neq \emptyset$ for every prime p and $U'(\mathbb{Q}) \neq \emptyset$, but $\mathscr{U}'(\mathbb{Z}[\frac{1}{17}]) = \emptyset$.

I.e, the Hasse principle for $\mathbb{Z}[\frac{1}{17}]$-valued points is violated. In particular, a failure of the integral Hasse principle occurs.

The violations are explained by a transcendental Brauer class.

A further modification of the example–Blowing up

Example

Let $\mathscr{S} \subset \mathrm{P}^3_{\mathbb{Z}}$ be given by the equation

$$-Y_0^2 Y_2 + Y_0 Y_1^2 + 2Y_0 Y_2^2 + Y_1 Y_2 Y_3 - 2Y_1^2 Y_2 + Y_2^2 Y_3 + Y_3^3 = 0$$

and put $\mathscr{V} := \mathscr{S} \backslash \mathscr{E}$, for the hyperplane $\mathscr{E} := V(Y_0) \subset \mathrm{P}^3_{\mathbb{Z}}$.

Then every integral point $(1 : y_1 : y_2 : y_3) \in \mathscr{V}(\mathbb{Z})$ such that $y_2 y_3 \neq 0$ satisfies $((y_2 - y_1^2)y_3, y_2)_2 = 1$ or $\gcd(2y_2 - 1, y_3) > 1$.

A further modification of the example–Blowing up

Example

Let $\mathscr{S} \subset \mathrm{P}^3_{\mathbb{Z}}$ be given by the equation

$$-Y_0^2Y_2 + Y_0Y_1^2 + 2Y_0Y_2^2 + Y_1Y_2Y_3 - 2Y_1^2Y_2 + Y_2^2Y_3 + Y_3^3 = 0$$

and put $\mathscr{V} := \mathscr{S} \backslash \mathscr{E}$, for the hyperplane $\mathscr{E} := V(Y_0) \subset \mathrm{P}^3_{\mathbb{Z}}$.

Then every integral point $(1\!:\!y_1\!:\!y_2\!:\!y_3) \in \mathscr{V}(\mathbb{Z})$ such that $y_2y_3 \neq 0$ satisfies $((y_2 - y_1^2)y_3, y_2)_2 = 1$ or $\gcd(2y_2 - 1, y_3) > 1$.

Proof. $\mathscr{S}$ is obtained from $\mathscr{X}$ by blowing up $(0:1:0:0:0)$. From the computational viewpoint, this means to eliminate X_1 from the equations defining $\mathscr{X}$. We replaced X_0, X_2, X_3, X_4 by Y_0, Y_1, Y_2, and Y_3.

An integral point on $\mathscr{V}$ is a $\mathbb{Q}$-rational point $(1:x_1:x_2:x_3:x_4) \in U(\mathbb{Q})$ such that x_2, x_3, and x_4 are integers, but x_1 not necessarily. If, however, $\gcd(2x_3-1, x_4) = 1$ then $x_4 \cdot x_1 = x_3 - x_2^2$ and $(2x_3-1) \cdot x_1 = -(x_4^2 + x_2x_3 + x_3^2)$ together imply that x_1 has to be an integer, as well. Then, $(x_1, x_3)_2 = 1$. □

Remark

There exist integral points on $\mathscr{V}$ of all three kinds allowed by the statement.

- For $(1\!:\!-1\!:\!2\!:\!-1)$, the gcd is 1 and the Hilbert symbol is 1.
- For $(1\!:\!15\!:\!-8\!:\!-17)$, the gcd is $17 > 1$ and the Hilbert symbol is 1.
- For $(1\!:\!5\!:\!2\!:\!3)$, the gcd is $3 > 1$ and the Hilbert symbol is (-1).

A further modification of the example–Blowing up III

Example

Let $\mathscr{S}' \subset \mathrm{P}^3_{\mathbb{Z}}$ be given by the equation

$$12Y_0^3 + 40Y_0^2Y_2 + 66Y_0^2Y_3 - 3Y_0Y_1^2 - 4Y_0Y_1Y_2 + 8Y_0Y_1Y_3 + 80Y_0Y_2^2 + 80Y_0Y_2Y_3 + 144Y_0Y_3^2 - 16Y_1^2Y_2 + 32Y_1Y_2Y_3 + 128Y_2^2Y_3 + 128Y_3^3 = 0$$

and put $\mathscr{V}' := \mathscr{S}' \backslash \mathscr{E}$, for the hyperplane $\mathscr{E} := V(Y_0) \subset \mathrm{P}^3_{\mathbb{Z}}$.

Then every integral point $(1 : y_1 : y_2 : y_3) \in \mathscr{V}'(\mathbb{Z})$ satisfies

$$\gcd(16y_2 + 3, 8y_3 + 3) > 1\,.$$

Proof. The equation is obtained from the example before by plugging in $(y_0, 2y_1 + 1, 16y_2 + 3, 8y_3 + 3)$ for (y_0, y_1, y_2, y_3). □

A further modification of the example–Blowing up IV

Remark

The surface $\mathscr{V}'$ contains infinitely many integral points. Indeed, define the two sequences c and c' in $\mathbb{Z}^3$ recursively by

$$c_1 := [-2,0,0], c_2 := [170,-24,-48], \quad c_{i+2} := -110c_{i+1} - c_i - [48,24,48],$$
$$c'_1 := [2,0,0], \quad c'_2 := [-266,-24,-48], c'_{i+2} := -110c'_{i+1} - c'_i - [48,24,48].$$

Then, for each $i \in \mathbb{N}$, $(1 : c_{i1} : c_{i2} : c_{i3}) \in \mathscr{V}'(\mathbb{Z})$ and $(1 : c'_{i1} : c'_{i2} : c'_{i3}) \in \mathscr{V}'(\mathbb{Z})$. [Intersection of $\mathscr{S}'$ with plane given by $Y_3 = 2Y_2$ contains $\mathbb{Q}$-rational (-1)-curve. Therefore splits off a conic. We solve a Pell-like equation.]

There are further integral points on $\mathscr{V}'$, for instance $(1 : 5414 : -803 : -1536)$ and $(1 : -344\,632 : 534 : 20\,706)$.

Moreover, both are the smallest members of infinite sequences of integral points of the same kind as above. The second member of the sequence starting at $(1 : 5414 : -803 : -1536)$ involves 1340-digit integers, already.

Another example

Example

Let the degree four del Pezzo surface $X \subset \mathrm{P}^4_{\mathbb{Q}}$ be given by the system of equations

$$(X_1 + X_4)X_4 = X_2^2 + (X_0 + X_4)^2\,,$$
$$(X_2 + X_4)(2X_2 + X_4) = 2X_1^2 + 3X_3^2\,.$$

and put $U := X \backslash H$ for $H := V(X_0)$.

Another example II

1. Then the manifold $X(\mathbb{R})$ consists of two connected components and its submanifold $U(\mathbb{R})$ decomposes into three connected components.
2. Strong approximation off $S_1 := \{p \text{ prime} \mid p \equiv 1 \pmod 4\}$ is violated. A $\mathbb{Z}[\frac{1}{S_1}]$-valued point must necessarily fulfil $(\frac{x_4}{x_0}, -1)_2 + (\frac{x_4}{x_0}, -1)_\infty = 0$, although not all adelic points outside S_1 satisfy this relation.
3. Similarly, there is a violation of strong approximation off
$$S_2 := \{p \text{ prime} \mid (\tfrac{-6}{p}) = 1\} = \{p \text{ prime} \mid p \equiv 1, 5, 7, 11 \pmod{24}\}\,.$$
A $\mathbb{Z}[\frac{1}{S_2}]$-valued point with $x_2 + x_4 \neq 0$ must necessarily fulfil the non-trivial relation $(\frac{x_2+x_4}{x_0}, -6)_2 + (\frac{x_2+x_4}{x_0}, -6)_3 + (\frac{x_2+x_4}{x_0}, -6)_\infty = 0$.
4. *In addition,* U is strongly obstructed at ∞, (according to our understanding in the non-connected case).
And therefore there is a violation of strong approximation off $\{\infty\}$.

Another example III

Idea of proof. Components:

- The sign of $\frac{X_2+X_4}{X_4}$ distinguishes the two components X_+ and X_-.
- $D := H \cap X$ does not meet X_-, but decomposes X_+ into two components according to the sign of $\frac{X_4}{X_0}$.

No transcendental Brauer classes:

- $D(\mathbb{Q}_3) = \varnothing \Longrightarrow D(\mathbb{Q}) = \varnothing$, but $J(D)(\mathbb{Q}) \cong \mathbb{Z}$.

Algebraic Brauer classes:

- The two rank-4 quadrics given yield the algebraic Brauer classes $\alpha_1, \alpha_2 \in \mathrm{Br}(U)$, given by $(\frac{x_4}{x_0}, -1)$ and $(\frac{x_2+x_4}{x_0}, -6)$.
- α_1 works only at 2 and ∞, while α_2 works only at 2, 3 and ∞.

Obstruction at ∞:

- X_- is a compact component of U. According to our definition, U is strongly obstructed at ∞.

Another example IV

Remark

- There is exactly one integral point on $\mathscr{U}$ lying on the compact component of $U(\mathbb{R})$, namely $(1{:}0{:}1{:}0{:}-1)$.
- A search for integral points on $\mathscr{U}$ delivered the following twelve others.

$(1 : 3 : 4 : \pm 13 : 17)$, $(1 : 147 : -452 : \pm 383 : 1409)$, $(1 : 12972 : 9043 : \pm 3550 : 6305)$, $(1 : 12759 : 15044 : \pm 20351 : 17741)$, $(1 : -2328 : -7367 : \pm 19622 : -23293)$, and $(1 : 2052 : 11143 : \pm 44472 : 60569)$.

Another example IV

Remark

- There is exactly one integral point on $\mathscr{U}$ lying on the compact component of $U(\mathbb{R})$, namely $(1:0:1:0:-1)$.
- A search for integral points on $\mathscr{U}$ delivered the following twelve others.

  $(1:3:4:\pm 13:17)$, $(1:147:-452:\pm 383:1409)$, $(1:12972:9043:\pm 3550:6305)$, $(1:12759:15044:\pm 20351:17741)$, $(1:-2328:-7367:\pm 19622:-23293)$, and $(1:2052:11143:\pm 44472:60569)$.

Violation of strong approximation off ∞:

- Consider the adelic point $\mathbbm{x}$ outside ∞ that is equal to $(1:0:1:0:-1)$ at every prime $p \neq 5, \infty$ and equal to $(1:3:4:13:17)$ at $p = 5$.

  Strong approximation off $\{\infty\}$ would imply that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of integral points $x_n \in \mathscr{U}(\mathbb{Z})$ being convergent to $\mathbbm{x}$ simultaneously with respect to the 2-, 3-, and 5-adic topologies.

  The Brauer classes α_1 and α_2 now enforce that x_n must be contained in the same connected component of $U(\mathbb{R})$ as the point $(-1:0:1:0:1)$. This component, however, does not contain any other integral point.

Another example V

In this example, algebraic Brauer classes interact with effects caused by $U(\mathbb{R})$ being disconnected into compact and non-compact components.

- The two non-compact components of $U(\mathbb{R})$ in fact fulfil the requirements of our definition of unobstructedness in the strong sense.
- Nevertheless, strong approximation off $\{\infty\}$ is violated, as there are Brauer classes α_1 and α_2 working at the primes 2, 3, and ∞.

Remark (concerning concepts of obstruction at ∞)

This shows that a serious definition of being unobstructed at infinity must include requirements on *all* connected components of $U(\mathbb{R})$.

Thanks

Thank you!!