# A solution to the inverse Galois problem for cubic surfaces

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joint work with

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Let $C \in \mathbf{P}^3$ be a smooth cubic surface over an algebraically closed field.

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- There is a pentahedron associated with general $C$ (Sylvester).
- There are (at least) two kinds of moduli spaces, coming out of the classical invariant theory.
  - The coarse moduli space of smooth cubic surfaces (Salmon, Clebsch).
  - The fine moduli space of *marked cubic surfaces* (Cayley, Coble).

# An arithmetic problem

Suppose the base field is $\mathbb{Q}$. Then the 27 lines are defined only over $\overline{\mathbb{Q}}$. $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ permutes them. The intersection pairing must be respected. I.e., after having fixed a marking on the lines, there is a homomorphism $\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to W(E_6)$.

**Definition**

One says that the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts upon the lines of $C$ via $G := \mathrm{im}\,\rho \subseteq W(E_6)$.

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**Problem**

Let $G \subseteq W(E_6)$ be any subgroup. Is there a smooth cubic surface $C$ over $\mathbb{Q}$ such that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts via $G$ upon the lines of $C$?

# The result

**Theorem (Elsenhans+J. 2012)**

*Let* $\mathfrak{g}$ *be an arbitrary conjugacy class of subgroups of* $W(E_6)$*.*

*Then there exists a smooth cubic surface* $C$ *over* $\mathbb{Q}$ *such that* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *acts on the lines of* $C$ *via a subgroup* $G \subseteq W(E_6)$ *belonging to the class* $\mathfrak{g}$*.*

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**Remark**

There are 350 conjugacy classes of subgroups in $W(E_6)$.

The result is thus proven as soon as we have a list of 350 examples.

These are available on our web pages, e.g.

`http://www.uni-math.gwdg.de/jahnel/linkstopapers.html`.

Actually, we offer six lists.

# How to verify the lists

This is the problem the other way round. Given a cubic surface such as

```
gl:= 5*x^3 - 9*x^2*y + 15*x^2*z - 216*x^2*w + 2*x*y*z + 5*x*y*w - 255*x*z^2 - 96*x*z*w + 104*x*w^2 +
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## Algorithm (Computation of the 27 lines and the Galois group)

1. Parametrize a line $\ell$ in the form $w = ay + bz, x = cy + dz$. The condition that $ell \subset C$ means

   $$(a:c:1:0), (b:d:0:1), ((a+b):(c+d):1:1), ((a-b):(c-d):1:(-1)) \in C.$$

   This defines an ideal $I \subset \mathbb{Q}[a, b, c, d]$ generated by four homogeneous cubic polynomials.

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2. Calculate a Gröbner base of $I$. Typically, this will consist of a univariate polynomial $p \in \mathbb{Q}[a]$ of degree 27 and formulas to compute $b, c, d$ from $a$.

   Otherwise, apply a random automorphism of $\mathbf{P}^3$ and go back to step 1.

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   Otherwise, apply a random automorphism of $\mathbf{P}^3$ and go back to step 1.

3. Calculate the Galois group of $p$. (Stauduhar's algorithm, Cannon-Holt) Output a list of generating permutations and the intersection matrix.

### Experiments

In a sample of 20000 random surfaces with coefficients in $[0 \dots 50]$, we found the whole $W(E_6)$, 20000 times.
The same for 20000 random surfaces with coefficients in $[-100 \dots 100]$.

$W(E_6)$ is the generic answer. To find a proper subgroup, a construction is necessary.

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## Remark

There are several very interesting constructions that work in particular cases. Nevertheless, the plan of this talk is to ignore about these and to present a construction that, in principle, works for an arbitrary subgroup $G \subseteq W(E_6)$.

# The pentahedral form

## Definition (Sylvester)

The cubic surface $S^{(a_0,\ldots,a_4)}$ given in $\mathbf{P}^4$ by

$$\begin{aligned} a_0X_0^3 + a_1X_1^3 + a_2X_2^3 + a_3X_3^3 + a_4X_4^3 &= 0\,, \\ X_0 + X_1 + X_2 + X_3 + X_4 &= 0\,, \end{aligned}$$

is said to be in *pentahedral form*.

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## Remarks

- The five planes given by $X_k = 0$ for $k = 0, \dots, 4$ form the *pentahedron* associated with the cubic surface.
- Over an algebraically closed field, every sufficiently general cubic surface has a unique pentahedron.
- If $a_0, \dots, a_4 \neq 0$ then the Hessian of $S^{(a_0,\dots,a_4)}$ is a quartic surface having exactly ten singular points. These are the triple intersection points of the five planes.

# The idea of explicit Galois descent

Suppose that $a_0, \dots, a_4$ are not in $\mathbb{Q}$, but the zeroes of a separable polynomial $f \in \mathbb{Q}[T]$ of degree 5. Then $S^{(a_0,\dots,a_4)}$ is, a priori, defined only over the splitting field $L$ of $f$.

## A construction

- Assume that $f$ is separable. Then $L' := \mathbb{Q}[T]/(f)$ is an étale algebra having exactly five embeddings $i_0, \dots, i_4 \colon L' \to L$. There is some $a \in L'$ such that $a_k = i_k(a)$, $k = 0, \dots, 4$.
- Choose a linear form $l \in L'[w, x, y, z]$ such that $l^{i_0} + \dots + l^{i_4} = 0$, but no further linear relations are true.

  (It turns out that this is fulfilled when the four coefficients of $l$ form a basis of the $L'$-vector space defined by $\operatorname{tr} x = 0$.)
- Then the cubic surface $S_{(a_0,\dots,a_4)}$ over $\mathbb{Q}$ defined by

$$\operatorname{Tr} al^3 = 0$$

  is $L$-isomorphic to $S^{(a_0,\dots,a_4)}$.

# The idea of explicit Galois descent II

## Remarks

- This construction immediately yields an algorithm. Computations have to be done in $L'$, but not in the Galois hull $L$.
- Substituting linear forms into the coordinates was a standard technique of the 19th century geometers.
- There is the general concept of Galois descent due to A. Weil, which was further generalized by A. Grothendieck. This requires a *descent datum* $\{U_\sigma\}_{\sigma \in G}$ for $G := \mathrm{Gal}(L/\mathbb{Q})$.

  Here, $U_\sigma \colon S^{(a_0,\ldots,a_4)} \to S^{(a_0,\ldots,a_4)}$ has to be a morphism twisted by $\sigma$ and the $U_\sigma$ together have to form a group operation.
- The construction above yields Galois descent for $U_\sigma = T_\sigma \circ \sigma$, where

$$\sigma \colon (x_0 : \ldots : x_4) \mapsto (\sigma(x_0) : \ldots : \sigma(x_4)),$$

$$T_\sigma \colon (x_0 : \ldots : x_4) \mapsto (x_{\Pi(\sigma)^{-1}(0)} : \ldots : x_{\Pi(\sigma)^{-1}(4)}),$$

  and $\Pi \colon G \to S_5$ is the permutation representation defined by the operation of $G$ on $a_0, \ldots, a_4$.

# Marked cubic surfaces

## Definition

Let $S$ be any scheme. Then a *family of cubic surfaces* over $S$ or a *cubic surface* over $S$ is a flat morphism $p\colon C \to S$ such that there exist

- a rank-4 vector bundle $\mathscr{E}$ on $S$,
- a non-zero section $c \in \Gamma(\mathscr{O}(3), \mathbf{P}(\mathscr{E}))$, and
- an isomorphism $\operatorname{div}(c) \stackrel{\cong}{\longrightarrow} C$ of $S$-schemes.

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## Definitions

1. A *line* on a smooth cubic surface $p \colon C \to S$ is a $\mathbf{P}^1$-bundle $l \subset C$ over $S$ such that, for every $x \in S$, one has $\deg_{\mathscr{O}(1)} l_x = 1$.

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2. A *family of marked cubic surfaces* over $S$ or a *marked cubic surface* over $S$ is a cubic surface $p\colon C \to S$ together with a sequence $(l_1, \dots, l_6)$ of six mutually disjoint lines.

   The sequence $(l_1, \dots, l_6)$ itself is called a *marking* of $C$.

# The moduli scheme of marked cubic surfaces

**Theorem**

*Let $K$ be a field.*

1. *Then there exists a fine moduli scheme $\widetilde{\mathscr{M}}$ for marked cubic surfaces over $K$. I.e., the functor*

$$F\colon \{K\text{-schemes}\} \longrightarrow \{\text{sets}\}\,,$$
$$S \mapsto \{\text{marked cubic surfaces over } S\}/\!\sim$$

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2. *$\widetilde{\mathscr{M}}$ is a smooth, quasi-projective fourfold and, in addition, a rational variety.*

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   *is representable by a $K$-scheme $\widetilde{\mathscr{M}}$.*

2. *$\widetilde{\mathscr{M}}$ is a smooth, quasi-projective fourfold and, in addition, a rational variety.*

**Idea of proof:** Let $\mathscr{U} \subset (\mathbf{P}^2)^6$ be the open subscheme parametrizing all ordered 6-tuples of points in $\mathbf{P}^2$ in general position. By the Hilbert-Mumford numerical criterion, $\mathscr{U}$ is contained in the $\mathrm{PGL}_3$-stable locus. One proves that $\mathscr{M} := \mathscr{U}/\,\mathrm{PGL}_3$ represents the functor $F$.

# The moduli scheme of marked cubic surfaces II

To give a point $p \in \mathscr{U}(\overline{K})$ is equivalent to giving $p_1, \ldots, p_6 \in \mathbf{P}^2(\overline{K})$ in general position. Projective geometry shows that there is a unique $\gamma \in \mathrm{PGL}_3(\overline{K})$ mapping $(p_1, p_2, p_3, p_4)$ to the standard projective basis $((1:0:0),(0:1:0),(0:0:1),(1:1:1))$.

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## Observation (A naive embedding)

The $\overline{K}$-rational points on $\widetilde{\mathscr{M}}$ may thus be represented by $3 \times 6$-matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & 1 & w & y \\ 0 & 1 & 0 & 1 & x & z \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Vanishing of the third coordinate of $p_5$ would mean that $p_1, p_2$, and $p_5$ were collinear, and similarly for $p_6$.

# The moduli scheme of marked cubic surfaces III

## Corollary

*We actually have an open embedding $\widetilde{\mathscr{M}} \hookrightarrow \mathbf{A}^4$. In particular, $\widetilde{\mathscr{M}}$ is an affine scheme.*

## Remark

The moduli scheme $\widetilde{\mathscr{M}}$ of marked cubic surfaces has its origins in the work of Arthur Cayley from 1849. His approach was as follows.

- There are 45 tritangent planes meeting the surface in three lines. Through each line there are five tritangent planes.
- This leads to a total of 135 cross ratios, which are invariants of the cubic surface, as soon as a marking is fixed on the lines. Only 45 of these cross ratios are essentially different, due to constraints within the cubic surfaces.
- They provide an embedding $\widetilde{\mathscr{M}} \hookrightarrow (\mathbf{P}^1)^{45}$. The closure of the image is Cayley's "cross ratio variety".

Cayley's "cross ratio variety" was rediscovered by I.Naruki in 1982.

# Coble's compactification—The gamma variety

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$\mathrm{SL}_3$ operates naturally on $\mathscr{O}(n)$, and hence on $\Gamma(\mathbf{P}^2, \mathscr{O}(n))$, for every $n$. But $\mathrm{PGL}_3$ does not. There is no $\mathrm{PGL}_3$-linearization for $\mathscr{O}_{\mathbf{P}^2}(1)$.

Via the canonical isogeny $\mathrm{SL}_3 \twoheadrightarrow \mathrm{PGL}_3$, the kernel of which consists of the multiples of the identity matrix by the third roots of unity, there is a canonical $\mathrm{PGL}_3$-linearization for $\mathscr{O}(3)$.

Further, the $\mathrm{PGL}_3$-invariant sections are the same as the $\mathrm{SL}_3$-invariant ones.

Minors of the $6 \times 3$-matrix

$$\begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} \\ x_{2,0} & x_{2,1} & x_{2,2} \\ & \dots & \\ x_{6,0} & x_{6,1} & x_{6,2} \end{pmatrix}$$

define invariant sections on $(\mathbf{P}^2)^6$.

# Coble's compactification—The gamma variety II

## Notation

- For example, for $1 \leq i_1 < i_2 < i_3 \leq 6$,

$$m_{i_1,i_2,i_3} := \det \begin{pmatrix} x_{i_1,0} & x_{i_1,1} & x_{i_1,2} \\ x_{i_2,0} & x_{i_2,1} & x_{i_2,2} \\ x_{i_3,0} & x_{i_3,1} & x_{i_3,2} \end{pmatrix}$$

  defines an invariant section of $\mathscr{O}(n_1) \boxtimes \ldots \boxtimes \mathscr{O}(n_6)$. Here, $n_i := 1$ when $i \in \{i_1, i_2, i_3\}$ and $n_i := 0$, otherwise.

  We write $m_{i_{\sigma(1)},i_{\sigma(2)},i_{\sigma(3)}} := m_{i_1,i_2,i_3}$ for $\sigma \in S_3$.

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  We write $m_{i_{\sigma(1)},i_{\sigma(2)},i_{\sigma(3)}} := m_{i_1,i_2,i_3}$ for $\sigma \in S_3$.

- Further,

$$d_2 := \det \begin{pmatrix} x_{1,0}^2 & x_{1,1}^2 & x_{1,2}^2 & x_{1,0}x_{1,1} & x_{1,0}x_{1,2} & x_{1,1}x_{1,2} \\ x_{2,0}^2 & x_{2,1}^2 & x_{2,2}^2 & x_{2,0}x_{2,1} & x_{2,0}x_{2,2} & x_{2,1}x_{2,2} \\ & & & \ldots & & \\ x_{6,0}^2 & x_{6,1}^2 & x_{6,2}^2 & x_{6,0}x_{6,1} & x_{6,0}x_{6,2} & x_{6,1}x_{6,2} \end{pmatrix}$$

  is an invariant section of $\mathscr{O}(2) \boxtimes \ldots \boxtimes \mathscr{O}(2)$.

# Coble's compactification—The gamma variety III

## Remark

One has the beautiful relation

$$d_2 = -\det \begin{pmatrix} m_{1,3,4}m_{1,5,6} & m_{1,3,5}m_{1,4,6} \\ m_{2,3,4}m_{2,5,6} & m_{2,3,5}m_{2,4,6} \end{pmatrix}.$$

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## Definition (Coble 1917)

For $\{i_1, \ldots, i_6\} = \{1, \ldots, 6\}$, consider

$$\gamma_{(i_1 i_2 i_3)(i_4 i_5 i_6)} := m_{i_1,i_2,i_3} m_{i_4,i_5,i_6}\, d_2 \qquad \text{and}$$

$$\gamma_{(i_1 i_2)(i_3 i_4)(i_5 i_6)} := m_{i_1,i_3,i_4} m_{i_2,i_3,i_4} m_{i_3,i_5,i_6} m_{i_4,i_5,i_6} m_{i_5,i_1,i_2} m_{i_6,i_1,i_2}$$

Following the original work, we call these 40 $SL_3$-invariant, and hence $PGL_3$-invariant, sections $\gamma_. \in \Gamma((\mathbf{P}^2)^6, \mathscr{O}(3) \boxtimes \ldots \boxtimes \mathscr{O}(3))$ the *irrational invariants*.

# Coble's compactification—The gamma variety IV

## Remarks

- Within the parentheses, the indices may be arbitrarily permuted without changing the symbol.

  Further, in the symbols $\gamma_{(i_1 i_2 i_3)(i_4 i_5 i_6)}$, the two triples may be interchanged.

  However, in the symbols $\gamma_{(i_1 i_2)(i_3 i_4)(i_5 i_6)}$, the three pairs may be permuted only cyclically. Thus, altogether there are ten invariants of the first type and 30 invariants of the second type.

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- The irrational invariants $\gamma_.$ do *not* generate the invariant ring

$$\bigoplus_{d \geq 0} \Gamma((\mathbf{P}^2)^6, \mathscr{O}(3d) \boxtimes \ldots \boxtimes \mathscr{O}(3d))^{\mathrm{PGL}_3}$$

  and do *not* define an embedding of the categorical quotient $((\mathbf{P}^2)^6)^{\text{semi-stable}} / \mathrm{PGL}_3$ into $\mathbf{P}^{39}$.

  They behave well, however, on $\mathscr{U} / \mathrm{PGL}_3 \subset ((\mathbf{P}^2)^6)^{\text{semi-stable}} / \mathrm{PGL}_3$.

# Coble's compactification—The gamma variety V

## Notation

The $\mathrm{PGL}_3$-invariant local sections of $\mathscr{O}(3) \boxtimes \ldots \boxtimes \mathscr{O}(3)$ form an invertible sheaf on $\widetilde{\mathscr{M}} = \mathscr{U}/\mathrm{PGL}_3$, which we will denote by $\mathscr{L}$.

## Theorem (Coble)

1. *The invertible sheaf $\mathscr{L}$ on $\widetilde{\mathscr{M}}$ is very ample.*
2. *The 40 irrational invariants $\gamma_{.} \in \Gamma(\widetilde{\mathscr{M}}, \mathscr{L})$ define a projective embedding $\gamma \colon \widetilde{\mathscr{M}} \hookrightarrow \mathbf{P}^{39}_K$.*
3. *The Zariski closure of the image of $\gamma$ is contained in a nine-dimensional linear subspace.*
4. *In this $\mathbf{P}^9$, it is the intersection of 30 cubic hypersurfaces.*

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## Definitions

- We call $\gamma\colon \widetilde{\mathscr{M}} \hookrightarrow \mathbf{P}^{39}_K$ *Coble's gamma map.*
- The Zariski closure of the image of $\gamma$ is called *Coble's gamma variety* and denoted by $\widetilde{M}$.

# Coble's compactification—The gamma variety VI

## Remarks

- We prove part 2, first. This implies part 1.
- To prove 2, we work in the naive embedding into $\mathbf{A}^4$. The gamma map is then given by 40 explicit polynomials. We show that they separate points and tangent vectors. (This requires some computer work.)
- Part 3 is equivalent to the statement that the vector space $\langle \gamma_. \rangle$ is only of dimension ten. This was known to Coble in 1917 and is easily checked by computer.
- Finally, the purely cubic expressions form a vector space of dimension 190. Since $\binom{12}{3} = 220$, this shows that there are 30 cubic relations, except for those coming from the linear ones.

  The intersection of the 30 cubic hypersurfaces in $\mathbf{P}^9$ is reported by `magma` as being reduced and irreducible of dimension four.

# The operation of $W(E_6)$

A marked cubic surface over $S$ is automatically smooth, according to our definition. All its 27 lines are defined over $S$. They may be labelled as $l_1, \dots, l_6$, $l'_1, \dots, l'_6$, $l''_{12}, l''_{13}, \dots, l''_{56}$.

There are exactly 51 840 possible markings for a smooth cubic surface with all 27 lines defined over the base. They are acted upon, transitively, by a group of that order, isomorphic to the Weyl group $W(E_6)$.

## Convention

We identify $W(E_6)$ with the permutation group acting on the 27 labels $l_1, \dots, l_6, l'_1, \dots, l'_6, l''_{12}, l''_{13}, \dots, l''_{56}$.

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$W(E_6)$ naturally operates on the moduli functor

$$F \colon \{K\text{-schemes}\} \to \{\text{sets}\} \, .$$

Therefore, it operates on the moduli scheme $\widetilde{\mathscr{M}}$.

# The operation of $W(E_6)$ II

Coble's (as well as Cayley's) compactifications explicitly linearize the operation of $W(E_6)$.

**Lemma (Coble)**

*There exists a $W(E_6)$-linearization of $\mathscr{L} \in \mathrm{Pic}(\widetilde{\mathscr{M}})$ such that*

- *the 80 sections $\pm\gamma_. \in \Gamma(\widetilde{\mathscr{M}}, \mathscr{L})$ form a $W(E_6)$-invariant set.*
- *The corresponding permutation representation $\Pi\colon W(E_6) \hookrightarrow S_{80}$ is transitive. It has a system of 40 blocks given by the pairs $\{\gamma, -\gamma\}$.*
- *The permutation representation $W(E_6) \hookrightarrow S_{40}$ on the 40 blocks is the same as that on decompositions of the 27 lines into three pairs of Steiner trihedra. It is also defined by the operation of $W(E_6)$ on its cosets modulo of its maximal subgroups of index* 40.

# The operation of $W(E_6)$ III

**Idea of proof:** We need a system of compatible isomorphisms $i_g \colon T_g^*\mathscr{L} \stackrel{\cong}{\longrightarrow} \mathscr{L}$ for $T_g \colon \widetilde{\mathscr{M}} \to \widetilde{\mathscr{M}}$ the operation of $g$.

For $g \in S_6 \subset W(E_6)$, there is an obvious such isomorphism defined by the permutation of the six labels.

$W(E_6)$ is generated by $S_6$ and one additional element, the quadratic transformation $I_{123}$. In the naive coordinates, this map is given by $(w, x, y, z) \mapsto (\frac{1}{w}, \frac{1}{x}, \frac{1}{y}, \frac{1}{z})$.

List explicit formulas for the 40 irrational invariants $\gamma_.$ in terms of these coordinates. Plugging in $(w, x, y, z) \mapsto (\frac{1}{w}, \frac{1}{x}, \frac{1}{y}, \frac{1}{z})$ in a naive way, yields an isomorphism $i'_{I_{123}} \colon T_{I_{123}}^*\mathscr{L} \stackrel{\cong}{\longrightarrow} \mathscr{L}$, permuting the 40 sections $\gamma_.$ up to signs and a common scaling factor of $\frac{1}{w^2x^2y^2z^2}$.

# Un-marked cubic surfaces

As cubic surfaces may have automorphisms, a fine moduli scheme cannot exist.

**Facts**

- *The quotient* $\widetilde{\mathscr{M}}/W(E_6) =: \mathscr{M}$ *is the coarse moduli scheme of smooth cubic surfaces.*
- *On the other hand,* $\mathscr{M} \cong \mathscr{V}/\mathrm{PGL}_4$ *for* $\mathscr{V} \subset \mathbf{P}^{19}$ *the open subscheme parametrizing smooth cubic surfaces.*

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## Remarks

- Every smooth cubic surface corresponds to a $\mathrm{PGL}_4$-stable point in $\mathbf{P}^{19}$.
- The $\mathrm{PGL}_4$-invariants have been determined by A. Clebsch as early as 1861. In today's language, Clebsch's result is that there is an open embedding $\mathrm{Cl}\colon \mathscr{V}/\mathrm{PGL}_4 \cong \mathscr{M} \hookrightarrow \mathbf{P}(1,2,3,4,5)$.

  One writes $A, \ldots, E$ for the coordinates of $\mathbf{P}(1,2,3,4,5)$.

# The pentahedral family

## Example

The *pentahedral* family $\mathscr{C} \to \mathbf{P}^4/S_5$ of cubic surfaces is given by

$$\begin{aligned} a_0X_0^3 + a_1X_1^3 + a_2X_2^3 + a_3X_3^3 + a_4X_4^3 &= 0\,, \\ X_0 + \quad X_1 + \quad X_2 + \quad X_3 + \quad X_4 &= 0\,. \end{aligned}$$

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## Remarks

1. One has $\mathbf{P}^4/S_5 \cong \mathbf{P}(1,2,3,4,5)$. The elementary symmetric functions $\sigma_1, \ldots, \sigma_5$ in $a_0, \ldots, a_4$ are natural homogeneous coordinates.
2. Restrict considerations to the open subset $\mathscr{P} \subset \mathbf{P}^4/S_5$ representing smooth cubic surfaces having a *proper pentahedron*. This means $\sigma_5 \neq 0$.

# The pentahedral family II

**Theorem (Clebsch/Salmon)**

*For* $t\colon \mathscr{P} \longrightarrow \mathscr{M}$ *the classifying morphism, the composition* $\mathrm{Cl}\circ t\colon \mathscr{P} \to \mathscr{M} \hookrightarrow \mathbf{P}(1,2,3,4,5)$ *is given by the* $S_5$*-invariant sections*

$$I_8 := \sigma_4^2 - 4\sigma_3\sigma_5, \quad I_{16} := \sigma_1\sigma_5^3, \quad I_{24} := \sigma_4\sigma_5^4, \quad I_{32} := \sigma_2\sigma_5^6, \quad I_{40} := \sigma_5^8$$

*of* $\mathscr{O}(8)$, $\mathscr{O}(16)$, $\mathscr{O}(24)$, $\mathscr{O}(32)$, *and* $\mathscr{O}(40)$, *respectively.*

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## Lemma

*The classifying morphism* $t\colon \mathscr{P} \to \mathscr{M}$ *is an open embedding.*

**Idea of proof:** This is not a deep observation. One shows actually that $\mathrm{Cl}\circ t\colon \mathscr{P} \to \mathbf{P}(1,2,3,4,5)$ is an open embedding. For that, one verifies that $\mathrm{Cl}\circ t$ is birational and finite, and that $\mathbf{P}(1,2,3,4,5)$ is a normal scheme.

# Coble's gammas versus Clebsch's invariants

## Theorem (Elsenhans+J. 2012)

1. *The canonical morphism*

$$\psi\colon \widetilde{\mathscr{M}} \xrightarrow{\text{pr}} \mathscr{M} \overset{\text{Cl}}{\hookrightarrow} \mathbf{P}(1,2,3,4,5)$$

*allows an extension to* $\mathbf{P}^{39}$ *under the gamma map. More precisely, there exists a rational map* $\widetilde{\psi}\colon \mathbf{P}^{39} \dashrightarrow \mathbf{P}(1,2,3,4,5)$ *such that the following diagram commutes,*

$$\begin{array}{ccccc} \widetilde{\mathscr{M}} & \xrightarrow{\text{pr}} & \mathscr{M} & \xrightarrow{\text{Cl}} & \mathbf{P}(1,2,3,4,5) \\ \gamma \downarrow & & & & \| \\ \mathbf{P}^{39} & & \overset{\widetilde{\psi}}{\dashrightarrow} & & \mathbf{P}(1,2,3,4,5)\,. \end{array}$$

# Coble's gammas versus Clebsch's invariants II

**Theorem (Elsenhans+J. 2012, continued)**

2. *Explicitly, the rational map* $\widetilde{\psi}\colon \mathbf{P}^{39} \dashrightarrow \mathbf{P}(1,2,3,4,5)$, *defined by the global sections*

   - $-6P_2 \in \Gamma(\mathbf{P}^{39}, \mathscr{O}(2))$,
   - $-24P_4 + \frac{41}{16}P_2^2 \in \Gamma(\mathbf{P}^{39}, \mathscr{O}(4))$,
   - $\frac{576}{13}P_6 - \frac{396}{13}P_4P_2 + \frac{29}{13}P_2^3 \in \Gamma(\mathbf{P}^{39}, \mathscr{O}(6))$,
   - $-\frac{62208}{1171}P_8 + \frac{54864}{1171}P_6P_2 + \frac{203616}{1171}P_4^2 - \frac{61287}{1171}P_4P_2^2 + \frac{13393}{4684}P_2^4 \in \Gamma(\mathbf{P}^{39}, \mathscr{O}(8))$,
   - $\frac{41472}{155}P_{10} - \frac{4605984}{36301}P_8P_2 - \frac{106272}{403}P_6P_4 + \frac{19990440}{471913}P_6P_2^2 + \frac{47719206}{471913}P_4^2P_2 - \frac{7468023}{471913}P_4P_2^3 + \frac{10108327}{18876520}P_2^5 \in \Gamma(\mathbf{P}^{39}, \mathscr{O}(10))$,

   *satisfies this condition. Here,* $P_k$ *denotes the sum of the 40* $k$*-th powers.*

3. *In other words, these formulas express Clebsch's invariants* $A, \ldots, E$ *in terms of Coble's 40 irrational invariants* $\gamma_.$.

# Coble's gammas versus Clebsch's invariants III

**Idea of proof:** $\psi :=$ Cl$\circ$pr defines a rational map $\varphi\colon \widetilde{M} \dashrightarrow \mathbf{P}(1,2,3,4,5)$ from the gamma variety. Extend $\psi$ to a morphism by closing the graph,

$$\begin{array}{ccc} \mathfrak{M} & & \\ {\scriptstyle \pi_1}\downarrow & \overset{\varphi'}{\searrow} & \\ \widetilde{M} & \overset{\varphi}{\dashrightarrow} & \mathbf{P}(1,2,3,4,5)\,. \end{array}$$

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One may show that $\varphi'^{*}\mathscr{O}(1) \cong \pi_1^{*}\mathscr{O}(2)|_{\widetilde{M}} \otimes \mathscr{O}(-E_1)$, where $E_1$ is an effective Cartier divisor supported in the exceptional fibers of $\pi_1$. Consequently,

$$\pi_{1*}\varphi^{*}\mathscr{O}(i) \subseteq \mathscr{O}(2i)|_{\widetilde{M}}\,.$$

The rational map $\varphi \colon \widetilde{M} \dashrightarrow \mathbf{P}(1,2,3,4,5)$ is therefore given by sections $t_i$ of $\mathscr{O}(2i)|_{\widetilde{M}}$, $i = 1, \ldots, 5$.

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It is classically known that $\varphi'^{-1}(A) = (-6) \sum_{j=0}^{39} X_j^2$. The other four sections extend to $\mathscr{O}(4), \ldots, \mathscr{O}(10)$ as the Castelnuovo-Mumford regularity of $\mathscr{I}_{\widetilde{M}}$ may be computed to 5.

# Coble's gammas versus Clebsch's invariants IV

The sections $t_i$ may be assumed $W(E_6)$-invariant. Molien's formula shows

$$\dim\Gamma(\mathbf{P}(V),\mathscr{O}(2i))^{W(E_6)} = \begin{cases} 1 & \text{for } i{=}1\,, \\ 2 & \text{for } i{=}2\,, \\ 5 & \text{for } i{=}3\,, \\ 11 & \text{for } i{=}4\,, \\ 23 & \text{for } i{=}5\,, \end{cases}$$

for $V$ the relevant 10-dimensional representation.

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for $V$ the relevant 10-dimensional representation.

All these sections may be found explicitly. The reduction process modulo a Gröbner base of $\mathscr{I}_{\widetilde{M}}$ then shows that

$$\Gamma(\widetilde{M}, \mathscr{O}(2i)|_{\widetilde{M}})^{W(E_6)} = \begin{cases} \langle P_2 \rangle & \text{for } i{=}1\,, \\ \langle P_4, P_2^2 \rangle & \text{for } i{=}2\,, \\ \langle P_6, P_4 P_2, P_2^3 \rangle & \text{for } i{=}3\,, \\ \langle P_8, P_6 P_2, P_4^2, P_4 P_2^2, P_2^4 \rangle & \text{for } i{=}4\,, \\ \langle P_{10}, P_8 P_2, P_6 P_4, P_6 P_2^2, P_4^2 P_2, P_4 P_2^3, P_2^5 \rangle & \text{for } i{=}5\,. \end{cases}$$

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To determine the coefficients in this basis is, finally, an interpolation problem.

# How to interpolate

## Algorithm (Pentahedron from cubic surface–Generic case)

1. Determine a Gröbner basis for the ideal $\mathscr{I}_{H_{\text{sing}}} \subset K[X_0, \ldots, X_3]$ of the singular locus of the Hessian $H$ of $C$. In particular, this yields a univariate degree-10 polynomial $\overline{F}$ defining an $S_5$-extension.
2. Uncover a degree-5 polynomial $F$ with the same splitting field.
3. Factorize $\overline{F}$ over $L$. Two irreducible factors, $\overline{F}_1$ of degree 4 and $\overline{F}_2$ of degree 6, are found.
4. Determine, in a second Gröbner base calculation, an element of minimal degree in the ideal $(\mathscr{I}_{H_{\text{sing}}}, \overline{F}_2) \subset L[X_0, \ldots, X_3]$. The result is a linear polynomial $l$. Its conjugates define the five individual planes that form the pentahedron.
5. Scale $l$ by a suitable non-zero factor from $L$ such that $\operatorname{Tr}_{L/K} l = 0$. Then calculate $a \in L$ such that the equation of the surface is exactly $\operatorname{Tr}_{L/K} al^3 = 0$.
6. Return $a$. One might want to return $l$ as a second value.

# Twisting Coble's gamma variety

Fix a continuous homomorphism $\rho\colon \mathrm{Gal}(\overline{K}/K) \to W(E_6)$ and consider

$$
\begin{aligned}
F_\rho\colon \{K\text{-schemes}\} &\longrightarrow \{\text{sets}\}\,, \\
S &\mapsto \big\{\text{marked cubic surfaces over } S_{\overline{K}} \text{ such that } \mathrm{Gal}(\overline{K}/K) \text{ operates on the 27 lines as described by } \rho\big\}/\!\sim\,,
\end{aligned}
$$

the moduli functor, *twisted by* $\rho$.

**Theorem (Elsenhans+J. 2012)**

*The functor $F_\rho$ is representable by a $K$-scheme $\widetilde{\mathscr{M}}_\rho$ that is a twist of $\widetilde{\mathscr{M}}$.*

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**Strategy (to construct a cubic surface for $G \subseteq W(E_6)$)**

1. First, one should find a Galois extension $L/\mathbb{Q}$ such that $\mathrm{Gal}(L/\mathbb{Q}) \cong G$. This defines the homomorphism $\rho$.
2. Then a $\mathbb{Q}$-rational point $P \in \widetilde{\mathscr{M}}_\rho(\mathbb{Q})$ is sought for.
3. For the corresponding cubic surface $\mathscr{C}_P$ over $\mathbb{Q}$, the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ operates on the 27 lines exactly as desired.

# The main algorithm

## Algorithm (Cubic surface for a given group)

Given a subgroup $G \subseteq W(E_6)$ and a field such that $\mathrm{Gal}(L/\mathbb{Q}) \cong G$, this algorithm computes a smooth cubic surface $C$ over $\mathbb{Q}$ such that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ operates upon the lines of $C$ via the group $\mathrm{Gal}(L/\mathbb{Q})$.

1. Fix a system $\Gamma \subseteq G$ of generators of $G$. For every $g \in \Gamma$, store the permutation $\Pi(g) \in S_{80}$, which describes the operation of $g$ on the 80 irrational invariants $\pm\gamma_.$. Further fix, once and for ever, ten of the $\pm\gamma_.$ that are linearly independent. Express the other 70 explicitly as linear combinations of these basis vectors.

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2. For every $g \in \Gamma$, determine the $10 \times 10$-matrix describing the operation of $g$ on the 10-dimensional $L$-vector space $\langle\gamma_.\rangle$. Use the explicit basis, fixed in 1.

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2. For every $g \in \Gamma$, determine the $10 \times 10$-matrix describing the operation of $g$ on the 10-dimensional $L$-vector space $\langle \gamma_. \rangle$. Use the explicit basis, fixed in 1.
3. Choose an explicit basis of the field $L$ as a $\mathbb{Q}$-vector space. Finally, make explicit the isomorphism $\rho^{-1} \colon G \rightarrow \mathrm{Gal}(L/\mathbb{Q}) \subseteq \mathrm{Hom}_{\mathbb{Q}}(L, L)$. I.e., write down a matrix for every $g \in \Gamma$.

# The main algorithm II

4. (Explicit Galois descent I) The condition that

$$(\sigma(x_{\Pi(\rho(\sigma))^{-1}(0)}), \ldots, \sigma(x_{\Pi(\rho(\sigma))^{-1}(79)})) = (x_0, \ldots, x_{79})$$

for all $\sigma \in G$ is a $\mathbb{Q}$-linear system of equations in $10[L : \mathbb{Q}]$ variables. We start with $\Gamma$ instead of $G$ and get $80[L : \mathbb{Q}]\#\Gamma$ equations. The result is a ten dimensional $\mathbb{Q}$-vector space $V \subset \langle \gamma_. \rangle$, described by an explicit basis.

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5. (Explicit Galois descent II)
   Convert the 30 cubic forms defining the image of $\gamma_L \colon \widetilde{\mathscr{M}_L} \hookrightarrow \mathbf{P}_L^{79}$ into terms of this basis of $V$. The result are 30 explicit cubic forms with coefficients in $\mathbb{Q}$. They describe the Zariski closure of $\widetilde{\mathscr{M}_\rho}$ in $\mathbf{P}(V)$.

# The main algorithm II

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$$(\sigma(x_{\Pi(\rho(\sigma))^{-1}(0)}), \dots, \sigma(x_{\Pi(\rho(\sigma))^{-1}(79)})) = (x_0, \dots, x_{79})$$

   for all $\sigma \in G$ is a $\mathbb{Q}$-linear system of equations in $10[L : \mathbb{Q}]$ variables. We start with $\Gamma$ instead of $G$ and get $80[L : \mathbb{Q}]\#\Gamma$ equations. The result is a ten dimensional $\mathbb{Q}$-vector space $V \subset \langle \gamma_. \rangle$, described by an explicit basis.

5. (Explicit Galois descent II)
   Convert the 30 cubic forms defining the image of $\gamma_L \colon \widetilde{\mathscr{M}_L} \hookrightarrow \mathbf{P}^{79}_L$ into terms of this basis of $V$. The result are 30 explicit cubic forms with coefficients in $\mathbb{Q}$. They describe the Zariski closure of $\widetilde{\mathscr{M}_\rho}$ in $\mathbf{P}(V)$.

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7. From the coordinates of the point found, read the 40 irrational invariants $\gamma_.$. Calculate Clebsch's invariants $A, \ldots, E$ from these. Then determine pentahedral coefficients and an explicit equation over $\mathbb{Q}$.

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- We used Galois-invariant geometric structures.
  - 158 classes stabilize a double-six,
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- The main algorithm had to be run only for the seven most complicated conjugacy classes.

Thank you!!