THE DISCRIMINANT OF A CUBIC SURFACE

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Abstract. The 27 lines on a smooth cubic surface over $\mathbb{Q}$ are acted upon by a finite quotient of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We construct explicit examples such that the operation is via the index two subgroup of the maximal possible group. This is the simple group of order 25 920. Our examples are given in pentahedral normal form with rational coefficients. For such cubic surfaces, we study the discriminant and show its relation to the index two subgroup.

1. Introduction

1.1. Let $\mathscr{S} \subset \mathbf{P}^3$ be a smooth cubic surface over an algebraically closed field. It is well known that there are exactly 27 lines on $\mathscr{S}$. Having numbered these lines appropriately, their intersection matrix is the same for every smooth cubic surface. The group of all permutations of the 27 lines respecting the intersection matrix is isomorphic to the Weyl group $W(E_6)$.

For a smooth cubic surface $S \subset \mathbf{P}^3$ over $\mathbb{Q}$, the 27 lines are, in general, not defined over $\mathbb{Q}$ but over an algebraic field extension L. The Galois group $\mathrm{Gal}(L/\mathbb{Q})$ is a subgroup of $W(E_6)$. It is known that equality holds for general cubic surfaces.

1.2. In this article, we describe our search for explicit examples of cubic surfaces over $\mathbb{Q}$ such that the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ is exactly the index two subgroup $D^1W(E_6) \subset W(E_6)$. This is the simple group of order 25 920.

Our approach is as follows. We consider cubic surfaces in pentahedral normal form with rational coefficients. For these, we study the discriminant Δ. We show that $\mathrm{Gal}(L/\mathbb{Q})$ is contained in the index two subgroup if and only if $(-3)\Delta$ is a perfect square. This leads to a point search on the double covering of $\mathbf{P}^4$, ramified at the degree 32 discriminantal variety.

A generalized Cremona transform reduces the degree to eight. We discuss the geometry of this modified discriminantal covering. In particular, we compute the Picard rank of a resolution of singularities. This has an application towards the arithmetic of the discriminantal covering, which we will investigate in a forthcoming paper [7].

2. The discriminant and the index two subgroup

2.1. One way to write down an explicit cubic surface is the so-called *pentahedral normal form.* Denote by $S^{(a_0,\dots,a_4)}$ the cubic surface given in $\mathbf{P}^4$ by the system of equations

$$\begin{aligned} a_0X_0^3 + a_1X_1^3 + a_2X_2^3 + a_3X_3^3 + a_4X_4^3 &= 0\,, \\ X_0 + \quad X_1 + \quad X_2 + \quad X_3 + \quad X_4 &= 0\,. \end{aligned}$$

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Remarks 2.2. a) A general cubic surface over an algebraically closed field may be brought into pentahedral normal form over that field. Further, the coefficients are unique up to permutation and scaling. This is a classical result, which was first observed by J. J. Sylvester [12]. A proof is given in [2]. Smooth cubic surfaces not allowing a pentahedral normal form correspond to a low-dimensional subset of the moduli stack [4, sec. 5].

Cubic surfaces over $\mathbb{Q}$ having a pentahedral normal form with rational coefficients are, however, special to a certain extent.

b) One should keep in mind that $S^{(0,a_1,\dots,a_4)}$ is simply the diagonal cubic surface with coefficients $a_1, \dots, a_4$.

Definition 2.3. The expression

$$\begin{aligned}&\Delta(S^{(a_0,\dots,a_4)}) := \\ &a_0^8 \cdot \ldots \cdot a_4^8 \cdot \\ &\prod_{i_1,i_2,i_3,i_4 \in \{0,1\}} \Big(\frac{1}{\sqrt{a_0}} + (-1)^{i_1} \frac{1}{\sqrt{a_1}} + (-1)^{i_2} \frac{1}{\sqrt{a_2}} + (-1)^{i_3} \frac{1}{\sqrt{a_3}} + (-1)^{i_4} \frac{1}{\sqrt{a_4}} \Big)\end{aligned}$$

is called the *discriminant* of the cubic surface $S^{(a_0,\dots,a_4)}$. Instead of $\Delta(S^{(a_0,\dots,a_4)})$, we will usually write $\Delta(a_0, \dots, a_4)$.

Remark 2.4. One has

$$\begin{aligned}&\Delta(a_0, \dots, a_4) := \\ &\prod_{i_1,i_2,i_3,i_4 \in \{0,1\}} (\sqrt{a_1a_2a_3a_4} + (-1)^{i_1} \sqrt{a_0a_2a_3a_4} + (-1)^{i_2} \sqrt{a_0a_1a_3a_4} + \cdots \\ &\qquad \cdots + (-1)^{i_3} \sqrt{a_0a_1a_2a_4} + (-1)^{i_4} \sqrt{a_0a_1a_2a_3})\,.\end{aligned}$$

Lemma 2.5. *$\Delta \in \mathbb{Q}[a_0, \dots, a_4]$ is a symmetric polynomial, homogeneous of degree 32, and absolutely irreducible.*

Proof. The remark shows that $\Delta \in \mathbb{Q}[\sqrt{a_0}, \dots, \sqrt{a_4}]$. Further, the expression is obviously invariant under the action of $G := \mathrm{Gal}(\mathbb{Q}(\sqrt{a_0}, \dots, \sqrt{a_4})/\mathbb{Q}(a_0, \dots, a_4))$. This yields $\Delta \in \mathbb{Q}[a_0, \dots, a_4]$. Symmetry and homogeneity are obvious.

Definition 2.3 provides us with the decomposition of Δ into irreducible factors in the unique factorization domain $\overline{\mathbb{Q}}[\sqrt{a_0}, \dots, \sqrt{a_4}, \frac{1}{a_0}, \dots, \frac{1}{a_4}]$. As G operates transitively on the sixteen factors, Δ is irreducible in $\overline{\mathbb{Q}}[a_0, \dots, a_4, \frac{1}{a_0}, \dots, \frac{1}{a_4}]$. Finally, from $\Delta(0, a_1, a_2, a_3, a_4) = (a_1a_2a_3a_4)^8$, we see that it is not divisible by a_0. Thus, Δ is not divisible by any of the a_i. $\square$

Lemma 2.6. *Writing σ_i for the elementary symmetric function of degree i in $a_0, \dots, a_4$, one may express the discriminant as follows,*

$$\Delta = (A^2 - 64B)^2 - 2^{11}(8D + AC)\,.$$

Here,

$$A := \sigma_4^2 - 4\sigma_3\sigma_5, \quad B := \sigma_1\sigma_5^3, \quad C := \sigma_4\sigma_5^4, \quad D := \sigma_2\sigma_5^6\,.$$

Proof. This formula may easily be established, for example, using `maple`. $\square$

Remarks 2.7. i) Together with $E := \sigma_5^8$, the expressions $A, B, C,$ and D are called the *fundamental invariants* of the cubic surface $S^{(a_0,\dots,a_4)}$. This notion is due to A. Clebsch [2].

ii) Lemma 2.6 is originally due to G. Salmon [11]. Note that there is a misprint in Salmon's original work, which has been repeatedly copied by several people throughout the 20th century. The correct formula may be found in [5].

Fact 2.8. *Assume that $a_0 \cdot \ldots \cdot a_4 \neq 0$. Then, the singular points on $S^{(a_0,\ldots,a_4)}$ are exactly those of the form*

$$\Big(\frac{1}{\sqrt{a_0}} : (-1)^{i_1}\frac{1}{\sqrt{a_1}} : (-1)^{i_2}\frac{1}{\sqrt{a_2}} : (-1)^{i_3}\frac{1}{\sqrt{a_3}} : (-1)^{i_4}\frac{1}{\sqrt{a_4}}\Big)$$

that lie on the hyperplane given by $X_0 + X_1 + X_2 + X_3 + X_4 = 0$. □

Examples 2.9. i) The cubic surface $S^{(1,1,1,1,\frac{1}{4})}$ has exactly four singular points. These are $(1 : -1 : -1 : -1 : 2)$ and permutations of the first four coordinates. This is the famous Cayley cubic.

ii) The cubic surface $S^{(1,1,1,\frac{1}{9},\frac{1}{16})}$ has exactly three singular points, namely $(1 : -1 : -1 : -3 : 4)$ and permutations of the first three coordinates.

iii) The cubic surface $S^{(1,1,\frac{1}{4},\frac{1}{9},\frac{1}{25})}$ has exactly two singular points. These are $(1 : -1 : -2 : -3 : 5)$ and permutations of the first two coordinates.

iv) $(-1 : -1 : -1 : -1 : 4)$ is the only singular point of the cubic surface $S^{(1,1,1,1,\frac{1}{16})}$.

Corollary 2.10. *The cubic surface $S^{(a_0,\ldots,a_4)}$ is non-singular if and only if $\Delta(a_0,\ldots,a_4) \neq 0$.* □

Remark 2.11. The same is true over any ground field of characteristic $\neq 3$. Therefore, with the possible exception of the prime 3, for $a_0,\ldots,a_4 \in \mathbb{Z}$ such that $\gcd(a_0,\ldots,a_4) = 1$, the prime divisors of $\Delta(a_0,\ldots,a_4)$ are exactly the primes, where $S^{(a_0,\ldots,a_4)}$ has bad reduction.

One might want to renormalize Δ in order to overcome the defect at the prime 3. For this, observe that $S^{(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3})}$ has only integral coefficients after the substitution $x_4 := -x_0 - \ldots - x_3$. It turns out that this surface has good reduction at 3. Since $\Delta(\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}) = -5 \cdot 3^{-27}$, actually $\pm 3^{27}\Delta(a_0,\ldots,a_4)$ could have the property desired. Theorem 2.12 below indicates that the minus sign should be correct.

Theorem 2.12. *Let $a_0,\ldots,a_4 \in \mathbb{Q}$ such that $\Delta(a_0,\ldots,a_4) \neq 0$. Then, the Galois group operating on the 27 lines on $S^{(a_0,\ldots,a_4)}$ is contained in the index two subgroup $D^1W(E_6) \subset W(E_6)$ if and only if $(-3)\Delta(a_0,\ldots,a_4) \in \mathbb{Q}$ is a perfect square.*

Proof. *First step.* Construction of a ramified covering of degree two of $\mathbf{P}^4$.

Define $\mathscr{C} \subset \mathbf{P}^4_{(X)} \times \mathbf{P}^4_{(x)}$ by the system of equations

$$\begin{aligned} x_0X_0^3 + x_1X_1^3 + x_2X_2^3 + x_3X_3^3 + x_4X_4^3 &= 0\,, \\ X_0 + \quad X_1 + \quad X_2 + \quad X_3 + \quad X_4 &= 0\,. \end{aligned}$$

The projection $\pi\colon \mathscr{C} \to \mathbf{P}^4 (= \mathbf{P}^4_{(x)})$ is the family of the cubic surfaces in pentahedral normal form. The fiber of π over $(x_0 : \ldots : x_4)$ is the cubic surface $S^{(x_0,\ldots,x_4)}$.

The fiber $\mathscr{C}_\eta$ over the generic point $\eta \in \mathbf{P}^4$ is a smooth cubic surface over $\mathbb{Q}(\eta) = \mathbb{Q}(x_1/x_0, x_2/x_0, x_3/x_0, x_4/x_0)$. Its 27 lines are defined over a finite extension L of $\mathbb{Q}(\eta)$. We claim that $\mathrm{Gal}(L/\mathbb{Q}(\eta)) = W(E_6)$.

Indeed, this is the maximal possible group. The inclusion "$\subseteq$" is, therefore, trivially fulfilled. On the other hand, according to a result of B. L. van der Waerden, the generic Galois group $\mathrm{Gal}(L/\mathbb{Q}(\eta))$ can not be smaller than that for a particular fiber. Specializing, for example, to $(x_0 : x_1 : x_2 : x_3 : x_4) = (1 : 2 : 3 : 7 : 17)$, [6, Algorithm 10] shows that the Galois group is equal to $W(E_6)$.

Consequently, there exists a unique intermediate field K of $L/\mathbb{Q}(\eta)$ that is quadratic over $\mathbb{Q}(\eta)$. This induces a scheme V together with a finite morphism $p\colon V \to \mathbf{P}^4$ of degree two.

In fact, this is a standard construction. For each affine open set $\operatorname{Spec} A = U \subseteq \mathbf{P}^4$, take the spectrum of the integral closure of A in the extension K. Note that A is integrally closed in $\mathbb{Q}(\eta)$ since $\mathbf{P}^4$ is a normal scheme. The morphism $p\colon V \to \mathbf{P}^4$ is finite according to the finiteness of the integral closure.

Second step. $p\colon V \to \mathbf{P}^4$ is unramified outside the divisor R given by $\Delta = 0$.

For this, let us describe the double covering V more precisely. We have $\mathbf{P}^3_{\mathbb{Q}(\eta)} \subset \mathbf{P}^4_{\mathbb{Q}(\eta)}$, given by the equation $X_0 + \ldots + X_4 = 0$, and a smooth cubic surface

$$\mathscr{C}_\eta \subset \mathbf{P}^3_{\mathbb{Q}(\eta)} \ .$$

On $\mathscr{C}_\eta$, there are the 45 tritangent planes. These give rise to a subscheme of the dual projective space $(\mathbf{P}^3)^\vee_{\mathbb{Q}(\eta)}$, which is finite of length 45 and étale over $\mathbb{Q}(\eta)$.

This, according to Galois theory, induces a set $M = \{e_1, \ldots, e_{45}\}$ of 45 elements together with an operation of $\operatorname{Gal}(\overline{\mathbb{Q}(\eta)}/\mathbb{Q}(\eta))$. Actually, only a finite quotient isomorphic to $W(E_6)$ is operating. The set M, in turn, defines a two element set $\{\pm e_1 \wedge ... \wedge e_{45}\}$, again acted upon by $\operatorname{Gal}(\overline{\mathbb{Q}(\eta)}/\mathbb{Q}(\eta))$. The fixgroup of this operation corresponds to the quadratic field extension $K/\mathbb{Q}(\eta)$.

We may do the same in a relative situation over $\mathbf{P}^4 \backslash R$. The 45 tritangent planes yield a closed subscheme of $(\mathbf{P}^3)^\vee \times (\mathbf{P}^4 \backslash R)$, which is finite and étale of degree 45 over $\mathbf{P}^4 \backslash R$. According to A. Grothendieck's theory of the étale fundamental group [8], this induces a set $M = \{e_1, \ldots, e_{45}\}$ of 45 elements together with an operation of $\pi_1^{\text{ét}}(\mathbf{P}^4 \backslash R, *)$. This group is canonically a quotient of $\operatorname{Gal}(\overline{\mathbb{Q}(\eta)}/\mathbb{Q}(\eta))$ [8, Exp. V, Proposition 8.2]. Again, we get a canonical operation on the two element set $\{\pm e_1 \wedge ... \wedge e_{45}\}$. Corresponding to this, there is an étale covering $p'\colon V' \to \mathbf{P}^4 \backslash R$ of degree two [8, Exp. V, Sec. 7].

V' is, by construction, a normal scheme with function field K. In particular, over an affine open set $\operatorname{Spec} A = U \subseteq \mathbf{P}^4 \backslash R$, we have the spectrum of the integral closure of A in the extension K. This shows that V and V' coincide over $\mathbf{P}^4 \backslash R$.

Third step. The equation.

As R is irreducible, the ramification locus of $p\colon V \to \mathbf{P}^4$ might be either empty or equal to R. If the ramification locus were empty then, as $\pi_1^{\text{ét}}(\mathbf{P}^4, *) = 0$, we had a trivial covering by a non-connected scheme. However, V is connected by construction. The generic fiber of p is a scheme consisting of a single point.

Hence, p is ramified exactly at R. This implies that V is given by the equation $w^2 = \lambda\Delta$ for a suitable constant λ.

Fourth step. Specialization.

Let $(a_0 : \ldots : a_4) \in \mathbf{P}^4(\mathbb{Q})$ such that $\Delta(a_0, \ldots, a_4) \neq 0$. Then, by virtue of the construction above, we have the following statement.

Denote by l the field of definition of the 27 lines on $S^{(a_0,\ldots,a_4)}$. Then, the smallest intermediate field k of $l/\mathbb{Q}$ such that $\operatorname{Gal}(l/k)$ acts on the 45 tritangent planes on $S^{(a_0,\ldots,a_4)}$ only via even permutations is exactly $k = \mathbb{Q}(\sqrt{\lambda\Delta(a_0, \ldots, a_4)})$.

This extension splits if and only if $\lambda\Delta(a_0, \ldots, a_4)$ is a perfect square in $\mathbb{Q}$. Except for the determination of the constant λ, this proves the assertion.

Fifth step. The constant λ.
We consider the particular cubic surface $S^{(0,1,1,1,1)}$, i.e., the diagonal cubic surface given by $x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$.

Here, the 45 tritangent planes are defined over the field $\mathbb{Q}(\zeta_3) = \mathbb{Q}(\sqrt{-3})$. They may easily be written down explicitly. In fact, seven of them are defined over $\mathbb{Q}$. Hence, $\mathrm{Gal}(\mathbb{Q}(\sqrt{-3})/\mathbb{Q})$ operates on the 45 tritangent planes as a product of 19 two-cycles, while seven tritangent planes are fixed. This is an odd permutation.

Consequently, in this case, $k = \mathbb{Q}(\sqrt{-3})$ is the smallest field such that $\mathrm{Gal}(l/k)$ acts on the 45 tritangent planes only by even permutations. As $\Delta(0,1,1,1,1) = 1$, this shows $\lambda = -3$ up to a factor being a perfect square. The proof is complete. □

Remark 2.13. This result was essentially known to H. Burkhardt [1, p. 341] in 1893. Burkhardt gives credit to C. Jordan [9], who was the first to study the automorphism group of the configuration of the 27 lines on a cubic surface.

3. Rational points on the discriminantal covering

Definition 3.1. We will call the twofold covering of $\mathbf{P}^4_{\mathbb{Q}}$, given by the equation

$$w^2 = -3\Delta(a_0, \ldots, a_4)\,, \tag{3.1}$$

the *discriminantal covering.*

3.2. There are two surprising constraints that equation (3.1) imposes on the coefficients $a_0, \ldots, a_4$.

Proposition 3.3 (The two constraints)**.** — *Suppose $a_0, \ldots, a_4 \in \mathbb{Z}$ are such that* $\gcd(a_0, \ldots, a_4) = 1$ *and* $(-3)\Delta(a_0, \ldots, a_4) \neq 0$ *is a perfect square in* $\mathbb{Q}$.

a) *Then, $a_0, \ldots, a_4$ all have the same sign.*

b) *Further, for every prime number* $p \equiv 2 \pmod 3$, *all the p-adic valuations $\nu_p(a_0), \ldots, \nu_p(a_4)$ are even.*

Proof. Observe first that the assumption ensures $a_0, \ldots, a_4 \neq 0$. Indeed, $\Delta(0, a_1, \ldots, a_4) = (a_1a_2a_3a_4)^8 \geq 0$.

a) Assume the contrary. Then, there is a product of four, say $a_1 \cdot \ldots \cdot a_4$, that is negative. The formula given in Remark 2.4 implies that $\Delta(a_0, \ldots, a_4)$ is the norm of an element of $\mathbb{Q}(\sqrt{a_1 \cdot \ldots \cdot a_4})$. As this is an imaginary quadratic field, we see that $\Delta(a_0, \ldots, a_4) \geq 0$. Contradiction!

b) Again, assume the contrary. Then, there is a product of four, say $a_1 \cdot \ldots \cdot a_4$, the p-adic valuation of which is odd. We have the fact that $\Delta(a_0, \ldots, a_4)$ is the norm of an element of $\mathbb{Q}(\sqrt{a_1 \cdot \ldots \cdot a_4})$. On the other hand, $(-3)\Delta(a_0, \ldots, a_4)$, being a perfect square by assumption, is a norm, too. Consequently, (-3) is the norm of an element of $\mathbb{Q}(\sqrt{a_1 \cdot \ldots \cdot a_4})$.

Since $\nu_p(a_1 \cdot \ldots \cdot a_4)$ is odd, the norm equation $(-3) = x^2 - a_1 \cdot \ldots \cdot a_4 \cdot y^2$ ensures that $\nu_p(x) = 0$ and $\nu_p(a_1 \cdot \ldots \cdot a_4 \cdot y^2) > 0$. Therefore, (-3) is a quadratic residue modulo p. This is a contradiction. □

3.4. We are interested in smooth cubic surfaces $S^{(a_0,\ldots,a_4)}$ such that the Galois group operating on the 27 lines is exactly equal to $D^1W(E_6)$.

By Theorem 2.12, this implies that $(a_0 : \ldots : a_4) \in \mathbf{P}^4(\mathbb{Q})$ gives rise to a $\mathbb{Q}$-rational point on the discriminantal covering. Further, according to Corollary 2.10, $(a_0 : \ldots : a_4)$ is supposed not to lie on the ramification locus.

Finally, if two of the coefficients were the same, say $a_0 = a_1$, then $S^{(a_0,\ldots,a_4)}$ allowed the tritangent plane $x_0+x_1 = 0$, which was defined over $\mathbb{Q}$. Consequently, the order of the group acting on the lines could be at most 1152.

3.5. A naive search. For these reasons, we searched for $\mathbb{Q}$-rational points $(w; a_0 : \ldots : a_4)$ satisfying equation (3.1) and the extra conditions below:

i) $w \neq 0$.

ii) No two of the five coordinates $a_0, \ldots, a_4$ are the same.

A rather simple computation led to the $\mathbb{Q}$-rational points $(3 : 4 : 21 : 36 : 63)$, $(4 : 7 : 12 : 28 : 84)$, and $(12 : 28 : 36 : 63 : 84)$. Up to symmetry, these are the only solutions of height ≤ 100.

Remark 3.6. The three rational points given above really lead to cubic surfaces such that the 27 lines are acted upon by the simple group $D^1W(E_6)$. To prove this, we ran the algorithm below, which is an obvious modification of [6, Algorithm 10].

Algorithm 3.7 (Verifying $G \supseteq D^1W(E_6)$)**.** — Given the equation $f = 0$ of a smooth cubic surface, this algorithm verifies that $G \subseteq W(E_6)$ is of index at most two.

i) Compute a univariate polynomial $0 \neq g \in \mathbb{Z}[d]$ of minimal degree such that

$$g \in (f(\ell(0)), f(\ell(\infty)), f(\ell(1)), f(\ell(-1))) \subset \mathbb{Q}[a, b, c, d]$$

where $\ell\colon t \mapsto (1 : t : (a + bt) : (c + dt))$.

If g is not of degree 27 then terminate with an error message. In this case, the coordinate system is not sufficiently general.

ii) Factor g modulo all primes below a given limit. Ignore the primes dividing the leading coefficient of g.

iii) If one of the factors is multiple then go to the next prime immediately. Otherwise, check whether the decomposition type is $(1, 1, 5, 5, 5, 5, 5)$ or $(9, 9, 9)$.

iv) If each of the two cases occurred at least once then output the message "The Galois group contains $D^1W(E_6)$." and terminate.

Otherwise, output "Cannot prove that the Galois group contains $D^1W(E_6)$."

Remark 3.8. To justify Algorithm 3.7, one may argue as follows. For $G \supseteq D^1W(E_6)$, it is sufficient to verify that G is transitive and contains an element of order five. Indeed, this criterion was proven in [6, Lemma 8].

Further, the two cycle types required guarantee these properties. An element of cycle type $(1, 1, 5, 5, 5, 5, 5)$ is of order five. Finally, G is transitive as neigher 9 nor 18 allows a partition into only fives and at most two ones.

4. The generalized Cremona transform

4.1. Δ is a homogeneous form of degree 32. Naively, one would expect that there are not many solutions of the equation

$$w^2 = -3\Delta(a_0, a_1, a_2, a_3, a_4)\,.$$

The constraints proven above reduce the expectations even more. Nevertheless, three rational points of height ≤ 100 have been found. The reason for this is the following observation.

Fact 4.2. *There is form Δ' homogeneous of degree 8 such that*

$$\Delta(a_0, \ldots, a_4) = (a_0 \cdot \ldots \cdot a_4)^8 \cdot \Delta'(1/a_0, \ldots, 1/a_4)\,.$$

Proof. The octic Δ' is given by the formula

$$\Delta'(x_0, \ldots, x_4) := \prod_{i_1,i_2,i_3,i_4 \in \{0,1\}} \left(\sqrt{x_0} + (-1)^{i_1}\sqrt{x_1} + (-1)^{i_2}\sqrt{x_2} + (-1)^{i_3}\sqrt{x_3} + (-1)^{i_4}\sqrt{x_4}\right). \;\square$$

Definition 4.3. We call the birational map ι from $\mathbf{P}^4$ to itself, given by

$$(a_0 : \ldots : a_4) \mapsto (1/a_0 : \ldots : 1/a_4)\,,$$

a *generalized Cremona transform.*

Remarks 4.4. i) Note that the standard Cremona transform of $\mathbf{P}^2$ is the birational map $(a_0 : a_1 : a_2) \mapsto (1/a_0 : 1/a_1 : 1/a_2)$.

ii) The generalized Cremona transform ι provides an automorphism of

$$\{(x_0 : \ldots : x_4) \in \mathbf{P}^4 \mid x_0 \cdot \ldots \cdot x_4 \neq 0\}\,.$$

In particular, $(x_0 : \ldots : x_4) \in \mathbf{P}^4(\mathbb{Q})$, $x_0 \cdot \ldots \cdot x_4 \neq 0$, gives rise to a solution of

$$w^2 = (-3)\Delta'(x_0, \ldots, x_4)$$

if and only if $\iota\big((x_0 : \ldots : x_4)\big)$ yields a rational point on the discriminantal covering.

Lemma 4.5. a) $\Delta' \in \mathbb{Q}[x_0, \ldots, x_4]$ *is a symmetric polynomial, homogeneous of degree eight and absolutely irreducible.*

b) *One has* $\Delta'(0, x_1, \ldots, x_4) = D^2$ *for a symmetric, homogeneous quartic form* $D \in \mathbb{Q}[x_1, \ldots, x_4]$.

Proof. a) By definition, $\Delta' \in \mathbb{Q}[\sqrt{x_0}, \ldots, \sqrt{x_4}]$. Further, the expression for Δ' is obviously invariant under the action of $G := \mathrm{Gal}(\mathbb{Q}(\sqrt{x_0}, \ldots, \sqrt{x_4})/\mathbb{Q}(x_0, \ldots, x_4))$. This yields $\Delta \in \mathbb{Q}[x_0, \ldots, x_4]$. Symmetry and homogeneity are obvious.

Finally, we have a decomposition of Δ' into irreducible factors in the unique factorization domain $\overline{\mathbb{Q}}[\sqrt{x_0}, \ldots, \sqrt{x_4}]$. Since G operates transitively on the sixteen factors, Δ is absolutely irreducible.

b) $\Delta'(0, x_1, \ldots, x_4)$ is the square of

$$D(x_1, \ldots, x_4) := \prod_{i_2,i_3,i_4 \in \{0,1\}} \left(\sqrt{x_1} + (-1)^{i_2}\sqrt{x_2} + (-1)^{i_3}\sqrt{x_3} + (-1)^{i_4}\sqrt{x_4}\right). \qquad \square$$

Remarks 4.6. i) The ramification locus R, given by $\Delta' = 0$ is a rational threefold. The parametrization $\iota\colon\ \mathbf{P}^3 \to R$,

$$\iota\colon (t_0 : \ldots : t_3) \mapsto \left(t_0^2 : t_1^2 : t_2^2 : t_3^2 : (t_0 + \ldots + t_3)^2\right)$$

is a finite birational morphism.

ii) The equation $D = 0$ defines the famous Roman surface of J. Steiner.

5. The Picard rank of the modified discriminantal covering

Definition 5.1. By the *modified discriminantal covering,* we mean the twofold covering O of $\mathbf{P}^4_{\mathbb{Q}}$, given by the equation

$$w^2 = -3\Delta'(a_0, \ldots, a_4).$$

Proposition 5.2. *The singular locus of O is reducible into ten components. The component $S_{(x_0,x_1)}$ is given by*

$$x_0 - x_1 = 0\,, \quad x_2^2 + x_3^2 + x_4^2 - 2x_2x_3 - 2x_2x_4 - 2x_3x_4 = 0\,.$$

The others are obtained by permuting coordinates.

Proof. *First case.* $x_0 \cdot \ldots \cdot x_4 \neq 0$.
Then, the morphism $p\colon \mathbf{P}^4_{\mathbb{Q}} \to \mathbf{P}^4_{\mathbb{Q}}$ given by

$$(t_0 : \ldots : t_4) \mapsto (t_0^2 : \ldots : t_4^2)$$

is étale over $(x_0 : \ldots : x_4)$. We may therefore test the fiber product $O \times_{\pi, \mathbf{P}^4_{\mathbb{Q}}, p} \mathbf{P}^4_{\mathbb{Q}}$ for smoothness. It is given explicitly by

$$w^2 = (-3) \prod_{i_1,i_2,i_3,i_4 \in \{0,1\}} \big(t_0 + (-1)^{i_1} t_1 + (-1)^{i_2} t_2 + (-1)^{i_3} t_3 + (-1)^{i_4} t_4\big)\,.$$

Here, the singular points are exactly the singular points of the ramification locus. That, in turn, consists of 16 hyperplanes such that precisely the intersection points are singular. Going back to O, we see that the singular points are those where at least two of the expressions

$$\sqrt{x_0} + (-1)^{i_1}\sqrt{x_1} + (-1)^{i_2}\sqrt{x_2} + (-1)^{i_3}\sqrt{x_3} + (-1)^{i_4}\sqrt{x_4}$$

vanish.

If these expressions coincide in one or four signs then this enforces one coordinate to be zero. The cases that there are two or three signs in common are essentially equivalent to each other. Without restriction,

$$\sqrt{x_0} - \sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} + \sqrt{x_4} = \sqrt{x_0} - \sqrt{x_1} - \sqrt{x_2} - \sqrt{x_3} - \sqrt{x_4} = 0\,.$$

Then, $\sqrt{x_0} = \sqrt{x_1}$ and $\sqrt{x_2} + \sqrt{x_3} + \sqrt{x_4} = 0$. The first equation yields $x_0 = x_1$. The quadratic relation given is equivalent to $\sqrt{x_2} \pm \sqrt{x_3} \pm \sqrt{x_4} = 0$.

Second case. $x_0 \cdot \ldots \cdot x_4 = 0$.
The singular locus is a Zariski closed subset. Therefore, the points satisfying the equations given above are clearly singular. It remains to prove that the others are non-singular.

Without restriction, we may assume that $x_0 = 0$ and that exactly one of the expressions $\sqrt{x_1} + (-1)^{i_2}\sqrt{x_2} + (-1)^{i_3}\sqrt{x_3} + (-1)^{i_4}\sqrt{x_4}$, say $\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} + \sqrt{x_4}$, is equal to zero. Then, the partial derivative of

$$\begin{aligned}(\sqrt{x_0} + \sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} + \sqrt{x_4})(\sqrt{x_0} - \sqrt{x_1} - \sqrt{x_2} - \sqrt{x_3} - \sqrt{x_4}) &= \\ = x_0 - (\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} + \sqrt{x_4})^2\end{aligned}$$

by x_0 is non-zero. As the other factors do not vanish, the product over all the 16 factors has non-zero derivative at this point. The assertion follows. $\square$

Theorem 5.3. *Let $\mathrm{pr}\colon \widetilde{O} \to O$ be the proper and birational morphism obtained by blowing up the ten singular components.*

a) *Then,* $\widetilde{O}$ *is non-singular. I.e.,* pr *is a resolution of singularities.*

b) *Further,* $\operatorname{rk}\operatorname{Pic}(\widetilde{O}) = 11$.

c) *The canonical divisor of* $\widetilde{O}$ *is* $K = \mathrm{pr}^*K_O$ *for* $K_O = -\pi^*H$ *and* H *a hyperplane section of* $\mathbf{P}^4$.

Proof. a) This may be tested locally. Let $(w; x_0 : \ldots : x_4)$ be a point in the singular locus of O.

First case. $x_0 \cdot \ldots \cdot x_4 \neq 0$.

Near $(x_0 : \ldots : x_4)$, the morphism

$$p\colon \mathbf{P}^4_{\mathbb{Q}} \to \mathbf{P}^4_{\mathbb{Q}}, \quad (t_0 : \ldots : t_4) \mapsto (t_0^2 : \ldots : t_4^2)$$

is étale. We may take square roots $t_0^{(0)}, \ldots, t_4^{(0)}$ of $x_0, \ldots, x_4$ and consider

$$w^2 = (-3) \prod_{i_1,i_2,i_3,i_4 \in \{0,1\}} \left(t_0 + (-1)^{i_1}t_1 + (-1)^{i_2}t_2 + (-1)^{i_3}t_3 + (-1)^{i_4}t_4\right).$$

Actually, only the linear factors vanishing at $(t_0^{(0)} : \ldots : t_4^{(0)})$ need to be taken into consideration.

Without restriction, suppose that $(x_0 : \ldots : x_4) \in S_{(x_0,x_1)}$. Then, again without restriction,

$$t_0^{(0)} - t_1^{(0)} + t_2^{(0)} + t_3^{(0)} + t_4^{(0)} = t_0^{(0)} - t_1^{(0)} - t_2^{(0)} - t_3^{(0)} - t_4^{(0)} = 0\,.$$

The corresponding linear forms X, Y are linearly independent, which means that we blow up a scheme, locally given by the equation $W^2 = XY$, at the ideal (X, Y). The result is clearly non-singular.

Now suppose that $(x_0 : \ldots : x_4)$ is a point of intersection of at least two singular components. Without loss of generality, the second singular component might be either $S_{(x_0,x_2)}$ or $S_{(x_2,x_3)}$. The latter variant enforces that $(x_0 : \ldots : x_4) = (1 : 1 : 1 : 1 : 4)$ is the point corresponding to the Cayley cubic. This is actually a special case of the first variant.

Thus, assume that $(x_0 : \ldots : x_4) \in S_{(x_0,x_1)} \cap S_{(x_0,x_2)}$. Then, without restriction, $t_0^{(0)} = t_1^{(0)} = t_2^{(0)}$ and $t_0^{(0)} + t_3^{(0)} + t_4^{(0)} = 0$. We have the three vanishing linear forms $t_0 + t_1 - t_2 + t_3 + t_4$, $t_0 - t_1 + t_2 + t_3 + t_4$, and $t_0 - t_1 - t_2 - t_3 - t_4$. Only when $x_3 = x_0$ (or $x_4 = x_0$), another linear form vanishes.

Altogether, there are four linearly independent linear forms X, Y, Z, and U. We blow up $W^2 = XYZU$ or $W^2 = XYZ$ at (X, Y), (X, Z), and (Y, Z), (as well as (X, U), (Y, U), and (Z, U)). The resulting scheme is non-singular.

Second case. Exactly one of the coordinates $x_0, \ldots, x_4$ vanishes.

Then, without loss of generality, $(x_0 : \ldots : x_4) = (a : a : b : b : 0)$. We may take square roots $t_0, \ldots, t_3$ of $x_0, \ldots, x_3$ such that t_0 and t_1 as well as t_2 and t_3 are of the same sign. Then, the right hand side goes over into the product over all $(t_0 \pm t_1 \pm t_2 \pm t_3)^2 - x_4$. Among these, $(t_0 - t_1 + t_2 - t_3)^2 - x_4$ and $(t_0 - t_1 - t_2 + t_3)^2 - x_4$ do vanish.

Hence, for two linearly independent linear forms X and Y, we consider the scheme given by $W^2 = (X^2 - x_4)(Y^2 - x_4)$. The singular components $S_{(x_0,x_1)}$ and $S_{(x_2,x_3)}$ correspond to the ideals $(X^2 - x_4, X + Y)$ and $(X^2 - x_4, X - Y)$, respectively. Blowing up the first ideal amounts to the substitutions $x_4 = X^2 + v(X + Y)$ and, for the other affine chart, $x_4 = X^2 + \frac{1}{v}(X + Y)$. The first substitution leads to $(W')^2 = v(X - Y + v)$ becoming smooth after blowing up $(v, X - Y)$, which is the next step. On the other hand, the second substitution yields $(W')^2 = v(X-Y)+1$ which is clearly non-singular near $v = 0$.

There is the exceptional case that $a = b$. Then, $(t_0 + t_1 - t_2 - t_3)^2 - x_4$ is a third factor vanishing. We have to consider a scheme locally given by $W^2 = (X^2 - x_4)(Y^2 - x_4)(Z^2 - x_4)$. Here, the substitution $x_4 = X^2 + v(X+Y)$ yields $(W')^2 = v(X - Y + v)[Z^2 - X^2 - v(X+Y)]$. The next step, to blow up $(v, X-Y)$, leads to $(W'')^2 = v_1(1+v_1)[Z^2 - X^2 - v_1(X^2 - Y^2)]$. Here, for the other affine chart, we find a formula of the same structure. Further, it is sufficient to consider the singularity at $v_1 = 0$. That at $v_1 = -1$ is analogous.

Actually, to blow up $S_{(x_1,x_2)} \cup S_{(x_0,x_3)}$ suffices to resolve this singularity. Indeed, the substitution $Z^2 - X^2 = v_2 v_1$ yields $(W''')^2 = v_2 - X^2 + Y^2$, which is clearly non-singular. On the other hand, putting $Z^2 - X^2 = \frac{1}{v_2} v_1$ leads to $(W''')^2 = v_2(1 - v_2(X^2 - Y^2))$, which is obviously smooth near $v_2 = 0$.

Third case. Exactly two of the coordinates $x_0, \ldots, x_4$ vanish.

Here, without restriction, $(x_0 : \ldots : x_4) = \big(0 : 0 : \big(t_2^{(0)}\big)^2 : \big(t_3^{(0)}\big)^2 : \big(t_4^{(0)}\big)^2\big)$ for $t_2^{(0)} + t_3^{(0)} + t_4^{(0)} = 0$. Therefore, precisely four of the sixteen factors of the right hand side vanish. These are $\sqrt{x_0} \pm \sqrt{x_1} \pm (t_2 + t_3 + t_4)$. We find $W^2 = X^2 - 2YT^2 + T^4$ for the new coordinate functions $X := x_0 - x_1$, $Y := x_0 + x_1$, and $T := t_2 + t_3 + t_4$.

When blowing up (X,T), the substitution $X := uT$ leads to $(W')^2 = u^2 - 2Y + T^2$, which is non-singular. On the other hand, $T := uX$ yields $(W')^2 = 1 - 2u^2 Y + u^4 X^2$ being clearly smooth near $u = 0$.

Fourth case. Three of the coordinates $x_0, \ldots, x_4$ vanish.

Without restriction, $(x_0 : \ldots : x_4) = (0 : 0 : 0 : 1 : 1)$. Take square roots t_3, t_4 of x_3 and x_4 that are of the same sign. The eight factors $\sqrt{x_0} \pm \sqrt{x_1} \pm \sqrt{x_2} \pm (t_3 - t_4)$ vanish at $(0 : 0 : 0 : 1 : 1)$. We find the local equation $W^2 = D(x_0, x_1, x_2, t^2)$ for $t := t_3 - t_4$ and D the symmetric, homogeneous quartic from Lemma 4.5.b).

Blowing up $(x_0 - x_1, x_2 - t^2)$ amounts to substituting $x_2 := t^2 + u(x_0 - x_1)$ and, for the other affine chart, $x_2 := t^2 + \frac{1}{u}(x_0 - x_1)$. Then, the ideals $(x_0 - x_2, x_1 - t^2)$ and $(x_1 - x_2, x_0 - t^2)$ to be blown up subsequently go over to $(u - 1, x_1 - t^2)$ and $(u + 1, x_0 - t^2)$. The substitutions

$$\begin{aligned} x_2 &:= t^2 + u(x_0 - x_1)\,, \\ x_1 &:= t^2 + u_1(u - 1)\,, \\ x_0 &:= t^2 + u_2(u + 1)\,, \end{aligned}$$

yield

$$W^2 = (Y - uZ)^2 + 8Zt^2$$

for the new functions $Y := u_1 + u_2$ and $Z := u_1 - u_2$.

For the other seven affine charts of this triple blow-up, the equations are completely analogous. The differences are that the definitions of Y and Z may be replaced by $Y, Z := 1 \pm u_1 u_2$. Further, instead of $Y - uZ$, we may have $uY - Z$.

The last step is to blow up the ideal $(Y - uZ, t)$ corresponding to the component $S_{(x_3,x_4)}$. The substitution $Y - uZ = vt$ yields $(W')^2 = v^2 + 8Z$, which is non-singular. Indeed, otherwise we must have $v = 0$ and $W' = 0$, which implies $Z = 0$. But, in this situation, Z is a local parameter. On the other hand, $Y - uZ = \frac{1}{v} t$ leads to $(W')^2 = 1 + 8v^2 Z$, which is clearly smooth at $v = 0$.

b) We claim that O is normal. To see this, note first that O is a hypersurface in weighted projective space $\mathbf{P} = \mathbf{P}(4,1,1,1,1,1)$. This is a scheme equipped with a canonical rational map $\iota\colon \mathbf{P} \dashrightarrow \mathbf{P}(1,1,1,1,1) = \mathbf{P}^4$. ι is undefined at exactly one point, which is the only singularity of $\mathbf{P}$.

By construction, the double covering O does not meet the singular point. Consequently, O is Gorenstein and, in particular, Cohen-Macaulay. Further, the singularities of O are in codimension 2. Serre's criterion [10, Theorem 23.8] shows that O is normal.

We assert that, after each step of blowing up, the resulting scheme is still normal. In fact, the centre of the blowing up is a codimension two complete intersection. The blow-up $\mathrm{Bl}_{S_{(x_0,x_1)}}(O)$ is, therefore, locally given by a single equation in a $\mathbf{P}^1$-bundle over O. This ensures $\mathrm{Bl}_{S_{(x_0,x_1)}}(O)$ is Cohen-Macaulay. Further, the smooth part of O is untouched under blowing up. Thus, regularity in codimension two could be destroyed only if the whole exceptional set were singular. As this is a $\mathbf{P}^1$-bundle over $S_{(x_0,x_1)}$, that is clearly not the case. The same argument works for each of the subsequent steps.

By Lemma 5.4, it suffices to show that the Picard rank grows by one in each step. Again, let us explain this for the first step in order to simplify notation. We have $\mathrm{Bl}_{S_{(x_0,x_1)}}(O) = \mathbf{Proj}(\mathscr{O} \oplus \mathscr{I} \oplus \mathscr{I}^2 \oplus \ldots)$ for $\mathscr{I} := \mathscr{I}_{S_{(x_0,x_1)},O}$. We assert that the twisting sheaf $\mathscr{O}(1)$ is linearly independent of the pull-backs of $\mathrm{Pic}(O)$ in $\mathrm{Pic}(\mathrm{Bl}_{S_{(x_0,x_1)}}(O))$. Indeed, $\mathscr{O}(n)$ for $n \neq 0$ is non-trivial when restricted to one of the exceptional fibers, which is just a $\mathbf{P}^1$.

c) As O is a Gorenstein scheme, its dualizing sheaf ω_O is invertible [3, Theorem 3.5.1]. To describe ω_O completely, we may restrict it to O^{reg}, since O is normal. Here, $\omega_O|_{O^{\mathrm{reg}}} \cong \Omega^4_{O^{\mathrm{reg}}}$. A 4-form with a simple pole at "$x_0 = 0$" is given by $(x_0^4/w) \cdot d(x_1/x_0) \wedge \ldots \wedge d(x_4/x_0)$. Hence, $\omega_O = \pi^*\mathscr{O}(-1)$.

Further, pr is an isomorphism outside the exceptional fibers. This implies that K and pr^*K_O coincide up to a sum of exceptional divisors. Due to symmetry, the coefficients at $E_1, \ldots, E_{10}$ are equal to each other. To determine the actual number, consider a general point $P \in S_{(x_0,x_1)}$. Near P, we blow up a double covering of the type $w^2 = XY$. This is a quadric cone times a neighbourhood of $(0,0) \in \mathbf{A}^2$. Its blow-up is the Hirzebruch surface Σ_2 times that neighbourhood. The exceptional curve $E \subset \Sigma_2$ is a (-2)-curve, hence $\omega_{\Sigma_2}|_E$ is trivial. The coefficients desired are equal to zero. □

Lemma 5.4. *Let $p\colon X \to Y$ be a surjective and birational morphism of Noetherian, normal, integral schemes. Then, the pull-back homomorphism $p^*\colon \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is injective.*

Proof. Suppose, for $\mathscr{L} \in \mathrm{Pic}(Y)$, the pull-back $p^*\mathscr{L} \in \mathrm{Pic}(X)$ would be trivial. This means, we have a section $s \in \Gamma(X, p^*\mathscr{L})$ without zeroes or poles. Corresponding to each codimension one point $\xi \in Y$, there is a discrete valuation ring $\mathscr{O}_\xi$. Further, there is a codimension one point $\zeta \in X$ mapping to ξ. As $\mathscr{O}_\xi$ is integrally closed, we see that $\mathscr{O}_\xi \cong \mathscr{O}_\zeta$.

Consequently, s gives rise to a section $t \in \Gamma(Y^\circ, \mathscr{L}|_{Y^\circ})$ without zeroes or poles for $Y^\circ \subseteq Y$ the complement of a closed subset of codimension ≥ 2. [10, Theorem 12.4.i)] implies that t may be extended to a global section. Hence, $\mathscr{L} \cong \mathscr{O}_Y$ is trivial. □

Remark 5.5. According to a famous conjecture of Manin, the value of the Picard rank of a non-singular Fano variety has implications on its arithmetic. More precisely, the number of points of anticanonical height $< B$ should grow asymptotically like $\tau B \log^{\operatorname{rk} \operatorname{Pic} \widetilde{O} - 1} B$ for a certain constant τ. We will discuss this topic in a forthcoming paper [7].

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