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Kummer surfaces and the computation of the Picard group

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Abstract

We test R. van Luijk's method for computing the Picard group of a $K3$ surface. The examples considered are the resolutions of Kummer quartics in $\mathbb{P}^3$. Using the theory of abelian varieties, the Picard group may be computed directly in this case. Our experiments show that the upper bounds provided by van Luijk's method are sharp when sufficiently many primes are used. In fact, there are a lot of primes that yield a value close to the exact one. However, for many but not all Kummer surfaces V of Picard rank 18, we have $\operatorname{rk}\operatorname{Pic}(V_{\overline{\mathbb{F}}_p}) \geqslant 20$ for a set of primes of density at least 1/2.

1. *Introduction*

1.1. For a general $K3$ surface V, the methods to date for computing the geometric Picard group are limited. As shown, for example, in [**5, 8, 18, 22**] or [**9**], it is possible to construct $K3$ surfaces of degree two or four with a prescribed Picard group; but when a $K3$ surface is given, say, by an equation with rational coefficients, it is not entirely clear whether its geometric Picard rank can be determined using currently available methods.

1.2. To be concrete, one can always establish a lower bound by specifying divisors explicitly and verifying that their intersection matrix is nondegenerate. On the other hand, for upper bounds, the method of R. van Luijk is available, which is based on reduction modulo p. It is not at all clear whether the upper bounds provided by van Luijk's method are always sharp.

Remark 1.3. It is conjectured that the Picard rank of a $K3$ surface over $\overline{\mathbb{F}}_p$ is always even. In particular, for V being a $K3$ surface over $\mathbb{Q}$, if $\operatorname{rk}\operatorname{Pic}(V_{\overline{\mathbb{Q}}})$ is odd, then there is no prime p of good reduction such that $\operatorname{rk}\operatorname{Pic}(V_{\overline{\mathbb{F}}_p}) = \operatorname{rk}\operatorname{Pic}(V_{\overline{\mathbb{Q}}})$. Furthermore, whether the rank over $\overline{\mathbb{Q}}$ is even or odd, there is no obvious reason why there should exist a prime number p such that $\operatorname{rk}\operatorname{Pic}(V_{\overline{\mathbb{F}}_p})$ is at least close to $\operatorname{rk}\operatorname{Pic}(V_{\overline{\mathbb{Q}}})$.

Definition 1.4. Let V be a $K3$ surface over $\mathbb{Q}$ and let p be a prime of good reduction. Then we will say that p is *good* if the geometric Picard rank of the reduction modulo p does not exceed the Picard rank over $\overline{\mathbb{Q}}$ by more than one.

1.5. In this article, we report on our experiments with van Luijk's method on a sample of Kummer surfaces. Kummer surfaces are particular $K3$ surfaces that allow a two-to-one covering by an abelian surface. The geometric Picard group of a Kummer surface is closely related to the Néron–Severi group of the abelian surface. In practice, it can be computed this way.

Nevertheless, for testing van Luijk's method, Kummer surfaces have big advantages. As the Picard ranks are known, the usual question of whether the lower bound or the upper bound needs to be improved does not arise. Further, using the special properties of a Kummer surface,

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one can massively optimize the point-counting step. In fact, it may very well be possible to compute $\operatorname{rk}\operatorname{Pic}(V_{\overline{\mathbb{F}}_p})$ for primes p up to 10 000.

1.6. Our sample consists of the resolutions of 9452 Kummer quartics with small coefficients. For each of these surfaces, we computed the upper bounds that were found using the primes $p \leqslant 997$. It turned out that good primes existed in every example. The upper bounds we found turned out to be equal to the geometric Picard ranks in all cases.

QUESTION 1.7. Do there exist good primes for every $K3$ surface over $\mathbb{Q}$?

1.8. *The method of van Luijk in detail*

The geometric Picard group of a $K3$ surface over $\mathbb{Q}$ is isomorphic to $\mathbb{Z}^n$ where n may range from 1 to 20. An upper bound for the geometric Picard rank can be computed as follows. One has the inequality

$$\operatorname{rk}\operatorname{Pic}(V_{\overline{\mathbb{Q}}}) \leqslant \operatorname{rk}\operatorname{Pic}(V_{\overline{\mathbb{F}}_p}),$$

which holds for every smooth variety V over $\mathbb{Q}$ and every prime p of good reduction. This is worked out in detail in [21, Remark 2.6.3], with the main input being [11, Example 20.3.6].

Further, for a $K3$ surface $\mathscr{V}$ over the finite field $\mathbb{F}_p$, one has the first Chern class homomorphism

$$c_1\colon\ \operatorname{Pic}(\mathscr{V}_{\overline{\mathbb{F}}_p}) \longrightarrow H^2_{\text{ét}}(\mathscr{V}_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l(1))$$

into l-adic cohomology. There is a natural operation of Frobenius on $H^2_{\text{ét}}(\mathscr{V}_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l(1))$. All eigenvalues are of absolute value 1. The Frobenius operation on the Picard group is compatible with the operation on cohomology.

Every divisor is defined over a finite extension of the ground field. Consequently, on the subspace

$$\operatorname{Pic}(\mathscr{V}_{\overline{\mathbb{F}}_p}) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_l \hookrightarrow H^2_{\text{ét}}(\mathscr{V}_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l(1)),$$

all eigenvalues are roots of unity. They correspond to eigenvalues of the Frobenius on $H^2_{\text{ét}}(\mathscr{V}_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l)$ being of the form $p\zeta$ for ζ a root of unity. One may therefore estimate the rank of the Picard group $\operatorname{Pic}(\mathscr{V}_{\overline{\mathbb{F}}_p})$ from above by counting how many eigenvalues are of this particular form.

Doing this for one prime, one obtains an upper bound for $\operatorname{rk}\operatorname{Pic}(V_{\overline{\mathbb{F}}_p})$ that is always even. The Tate conjecture asserts that this bound is actually sharp. For this reason, one tries to combine information from two primes. The assumption that the surface would have Picard rank $2r$ over $\overline{\mathbb{Q}}$ and $\overline{\mathbb{F}}_p$ implied that the discriminants of both Picard groups, $\operatorname{Pic}(V_{\overline{\mathbb{Q}}})$ and $\operatorname{Pic}(V_{\overline{\mathbb{F}}_p})$, were in the same square class. Note here that reduction modulo p respects the intersection product. When combining information from two primes, it may happen that one finds the rank bound $2r$ twice but the square classes of the discriminants are not the same. Then, these data are incompatible with Picard rank $2r$ over $\overline{\mathbb{Q}}$. There is a rank bound of $(2r-1)$.

REMARK 1.9. Some refinements of the method of van Luijk are described in [8] and [9]. We will not test these refinements here.

EXAMPLE 1.10. Let V be a $K3$ surface of geometric Picard rank one. We denote by

$$V^n := \bigtimes_{i=1}^{n} V$$

the n-fold cartesian product. Then, the Picard rank of V^n is equal to n.

Indeed, as we have $H^1(V(\mathbb{C}),\mathbb{Z})=0$, the Künneth formula shows that $H^2(V(\mathbb{C})^n,\mathbb{Z})\cong H^2(V(\mathbb{C}),\mathbb{Z})^n$. There is an analogous isomorphism for cohomology with complex coefficients, which is compatible with Hodge structures. That is, $H^{1,1}(V(\mathbb{C})^n)\cong H^{1,1}(V(\mathbb{C}))^n$. The assertion now follows from the Lefschetz theorem on $(1,1)$-classes; see [**12**, p. 163].

Assuming the Tate conjecture, one sees that the Picard rank of the reduction of V^n at an arbitrary prime is at least $2n$. This shows that there is no good prime. Not knowing the decomposition of V^n into a direct product, we could not determine its Picard rank.

CONVENTION 1.11. *Let V be a projective variety over a field k. In this article, unless stated otherwise, the Picard rank of V will always mean the geometric Picard rank, that is, the rank of* $\mathrm{Pic}(V_{\overline{k}})$.

The analytic discriminant and the Artin–Tate formula

For the final step in 1.8, one needs to know the discriminant of the Picard lattice. One possible way to compute this is to use the Artin–Tate formula.

CONJECTURE 1.12 (Artin–Tate). *Let V be a $K3$ surface over a finite field $\mathbb{F}_q$. Denote by ρ the rank and by Δ the discriminant of the Picard group of V, defined over $\mathbb{F}_q$. Then*

$$|\Delta|=\frac{\lim\limits_{T\to q}\frac{\Phi(T)}{(T-q)^\rho}}{q^{21-\rho}\#\mathrm{Br}(V)}.$$

Here, Φ denotes the characteristic polynomial of Frob *on $H^2_{\text{ét}}(V_{\overline{\mathbb{F}}_q},\mathbb{Q}_l)$. Finally,* $\mathrm{Br}(V)$ *is the Brauer group of V.*

REMARKS 1.13. (i) The Artin–Tate conjecture was first formulated, more generally than just for $K3$ surfaces, as Conjecture (C) in [**30**, p. 426].

(ii) Conjecture 1.12 has been proved for most $K3$ surfaces. Most notably, the Tate conjecture implies the Artin–Tate conjecture [**23**, Theorem 6.1]. In these cases, $\#\mathrm{Br}(V)$ is a perfect square. On its part, the Tate conjecture is proven for $K3$ surfaces under various additional assumptions. For example, it is true for elliptic $K3$ surfaces [**1**]. For ordinary $K3$ surfaces, it is known to be true as well [**28**], but we will not need this fact.

(iii) The Artin–Tate formula allows us to compute the square class of the discriminant of the Picard group over a finite field. No knowledge of explicit generators is necessary.

2. *Singular quartics*

Singular quartic surfaces were extensively studied by the geometers of the 19th century, particularly A. Cayley and E. E. Kummer. For example, the concept of a trope was introduced at that time [**17**].

DEFINITION 2.1. Let $Q\subset\mathbb{P}^3$ be any quartic surface. Then, by a *trope* on Q we mean a plane E such that $Q\cap E$ is a double conic. This is equivalent to the condition that the equation defining Q becomes a perfect square on E.

REMARK 2.2. A trope yields a singular point on the surface $Q^\vee\subset(\mathbb{P}^3)^\vee$ dual to Q.

LEMMA 2.3 (Kummer). *A quartic surface without singular curves may have at most* 16 *singular points.*

A classical family. A classification of the singular quartic surfaces with at least eight singularities of type A_1 was given by K. Rohn [**29**]; cf. [**17**, Chapter I]. In this article, we deal with one of the most important classical families.

LEMMA 2.4 (Kummer). *A three-dimensional family of quartics in $\mathbb{P}^3$ such that the generic member has exactly* 16 *singularities of type A_1 and no others is given by the equation*

$$16kxyzw - \phi^2 = 0.$$

Here

$$k := a^2 + b^2 + c^2 - 1 - 2abc,$$
$$\phi := x^2 + y^2 + z^2 + w^2 + 2a(yz + xw) + 2b(xz + yw) + 2c(xy + zw)$$

for parameters a, b, and c.

REMARKS 2.5. (i) E. E. Kummer introduced this family in Section 10 of his report [19].

(ii) We will write $Q_{[a,b,c]}$ for the quartic corresponding to the triple $[a, b, c]$. Up to isomorphism, this surface is independent of the order of a, b and c. Further, there is the isomorphism $Q_{[a,b,c]} \xrightarrow{\cong} Q_{[-a,-b,c]}$ given by $(x : y : z : w) \mapsto ((-x) : (-y) : z : w)$.

(iii) When one of the coefficients is equal to ± 1, $Q_{[a,b,c]}$ contains a singular line. For example, the surfaces for $a = \pm 1$ contain the singular line given by $x + aw = y + az = 0$.

(iv) On the generic fiber, there are twelve obvious singularities defined over quadratic extensions of $\mathbb{Q}(a, b, c)$. These are given by $x = y = 0$, $z^2 + w^2 + 2czw = 0$, and the analogous conditions with the roles of the variables interchanged. Further, there are four singular points forming a Galois orbit.

(v) On a Kummer quartic, there are 16 tropes. Four of them are obvious: they are explicitly given by the coordinate planes. Each trope passes through six of the 16 singular points, and each singular point is contained in six tropes [14, Chapter I].
On an obvious trope, the conic has discriminant $2abc + 1 - a^2 - b^2 - c^2 = -k$. Thus, these conics are nondegenerate except for the case where Q is non-reduced itself.

(vi) For a generic Kummer quartic, every singular point on $Q^\vee$ comes from a trope.

3. *The desingularization*

LEMMA 3.1. *Let $\pi\colon \widetilde{Q} \to Q$ be the desingularization of a normal quartic surface Q such that all singularities are of type A_1. Then $\widetilde{Q}$ is a $K3$ surface.*

Proof. On the smooth part of Q, the adjunction formula [12, Section 1.1, Example 3] may be applied as usual. Because for the canonical sheaf one has $K_{\mathbb{P}^3} = \mathscr{O}(-4)$, this shows that the invertible sheaf $\Omega^2_{Q^{\mathrm{reg}}}$ is trivial. Consequently, $K_{\widetilde{Q}}$ is given by a linear combination of the exceptional curves.

However, for an exceptional curve E, we have $E^2 = -2$. Hence, according to the adjunction formula, $K_{\widetilde{Q}}E = 0$, which shows that $K_{\widetilde{Q}}$ is trivial. The classification of algebraic surfaces [2] ensures that $\widetilde{Q}$ is either a $K3$ surface or an abelian surface.

Further, a standard application of the theorem on formal functions implies $R^1\pi_*\mathscr{O}_{\widetilde{Q}} = 0$. Hence, $\chi_{\mathrm{alg}}(\widetilde{Q}) = \chi_{\mathrm{alg}}(Q) = 2$. This shows that $\widetilde{Q}$ is actually a $K3$ surface. □

REMARKS 3.2. (i) For the assertion of the lemma, it is actually sufficient to assume that the singularities of Q are of types A, D, or E; see [20].

(ii) In general, the desingularization of a normal quartic surface is a $K3$ surface, a rational surface, a ruled surface over an elliptic curve, or a ruled surface over a curve of genus three [16]. The last possibility is caused by a quadruple point. The existence of a triple point implies that the surface is rational. It is, however, also possible that there is a double point not of type A, D, or E. In that case, $\widetilde{Q}$ is rational or a ruled surface over an elliptic curve.

LEMMA 3.3. *Let $\pi\colon \widetilde{Q} \to Q$ be the desingularization of a proper surface Q having only A_1-singularities. Then:*

(a) *the exceptional curves define a nondegenerate orthogonal system in* $\mathrm{Pic}(\widetilde{Q})$;
(b) *in particular, the Picard rank of* $\widetilde{Q}$ *is strictly greater than the number of singularities of* Q.

Proof. (a) The exceptional curves have self-intersection number (-2) and do not meet each other.

(b) For H being the hyperplane section, $\pi^*\mathscr{O}_Q(H)$ is orthogonal to the exceptional curves. □

4. *Abelian surfaces and Kummer quartics*

Let A be an abelian surface. Denote by $\phi\colon A \to A$ the involution given by $p \mapsto (-p)$. Then, the quotient $A/\sim$ for $\sim := \{(p, \phi(p)) \mid p \in A\}$ has precisely 16 singular points. We call such a quotient an *abstract Kummer surface.*

FACT 4.1. Let A be an abelian surface over a field k of characteristic zero and let V be the resolution of the corresponding Kummer surface. Then $\mathrm{rk}\,\mathrm{Pic}(V_{\overline{k}}) = \mathrm{rk}\,\mathrm{NS}(A_{\overline{k}}) + 16$.

Proof. A standard argument [**13**, Proposition (8.9.1)] allows us to assume that k is finitely generated over $\mathbb{Q}$. Then, in particular, k allows an embedding into $\mathbb{C}$. The canonical injection $\iota\colon H^2(V(\mathbb{C}), \mathbb{Z}) \to H^2(A(\mathbb{C}), \mathbb{Z})$ yields a bijection of $H^2(V(\mathbb{C}), \mathbb{Z})$ with $\langle E_1, \ldots, E_{16}\rangle^\perp$. As ι respects the $(1, 1)$-classes, the assertion follows. Observe that a base change to $\mathbb{C}$ does not change the Picard and Néron–Severi ranks. □

LEMMA 4.2 (Nikulin). *Let* Q *be a quartic surface over an algebraically closed field* k *of characteristic zero with precisely* 16 *singular points of type* A_1 *and no others. Then* Q *is isomorphic to an abstract Kummer surface.*

Proof. This result is shown in [**27**]. We include a sketch of the proof for the reader's convenience.

Again, we may assume that k is a subfield of $\mathbb{C}$. As shown in Lemma 3.1, the desingularization $\widetilde{Q}$ is a $K3$ surface. We have to prove that $\widetilde{Q}$ admits a double cover ramified exactly at the 16 exceptional curves $E_1, \ldots, E_{16}$. This is equivalent to the assertion that $\mathscr{O}(E_1 + \ldots + E_{16}) \in \mathrm{Pic}(\widetilde{Q})$ is divisible by two.

Consider, more generally, the set C of all $\mathbb{Q}$-divisors $D = c_1E_1 + \ldots + c_{16}E_{16}$ that define an element of $\mathrm{Pic}(\widetilde{Q})$. Clearly, $c_1, \ldots, c_{16} \in \frac{1}{2}\mathbb{Z}$, as otherwise the intersection numbers with $E_1, \ldots, E_{16}$ would not be integers. Thus, C defines a sub-vector space $\overline{C}$ of

$$\bigoplus_{i=1}^{16} \tfrac{1}{2}\mathbb{Z}E_i \Big/ \bigoplus_{i=1}^{16} \mathbb{Z}E_i \cong \mathbb{F}_2^{16}.$$

We claim that $\dim \overline{C} \geqslant 5$. Indeed, if not, then the lattice $C \subset \mathrm{Pic}(\widetilde{Q})$ would have a basis containing twelve of the standard elements $E_1, \ldots, E_{16}$. As the quotients $H^2(\widetilde{Q}(\mathbb{C}), \mathbb{Z})/\mathrm{Pic}(\widetilde{Q})$ and $\mathrm{Pic}(\widetilde{Q})/C$ have no torsion, $H^2(\widetilde{Q}(\mathbb{C}), \mathbb{Z})$ would still have a basis containing twelve of the E_i. But then the 22×22 matrix of the cup-product form would contain a symmetric 12×12 block consisting entirely of even entries. This ensures that the determinant is even and, hence, is a contradiction to the unimodularity of $H^2(\widetilde{Q}(\mathbb{C}), \mathbb{Z})$.

Further, every vector in $\overline{C}$ is a sum of precisely eight or 16 standard basis vectors. In fact, if it is a sum of l basis vectors, then it defines a double cover P' of $\widetilde{Q}$ ramified at exactly l of the 16 exceptional curves $E_1, \ldots, E_{16}$. Its minimal model P, obtained by blowing down the l exceptional curves, clearly has trivial canonical class. It is therefore either an abelian surface, $\chi_{\text{top}}(P) = 0$, or a $K3$ surface, $\chi_{\text{top}}(P) = 24$. But a direct calculation shows that $\chi_{\text{top}}(P) = 48 - 3l$.

Finally, it is a well-known result from coding theory [**15**, Theorem 2.7.4] that there is no five-dimensional subspace of $\mathbb{F}_2^{16}$ such that every nonzero vector has exactly eight components equal to 1. Indeed, adding the vector $(1, 1, \ldots, 1)$ would yield a code that contradicts the optimality of the $[16, 5, 8]$-Hadamard code. $\square$

4.3. Consider the particular case where $A = J(C)$ is the Jacobian of a curve C of genus two. Then, a projective model of the corresponding Kummer surface may be obtained as follows.

For a Weierstraß point r of C, put $\theta := \{[x] - [r] \mid x \in C\} \subset J(C)$. This is an ample divisor on the Jacobian $J(C)$ such that $\theta^2 = 2$. The Riemann–Roch theorem shows that $\dim \Gamma(J(C), 2\theta) = 4$. Hence, 2θ defines a morphism $\iota\colon J(C) \to \mathbb{P}^3$ of degree eight. Actually, ι is a two-to-one map that induces an embedding of $J(C)/\sim$ (see [**2**, Chapter VIII, Exercise 4]). The image of ι is a quartic surface.

4.4. It is a classical result that every Kummer quartic Q can be constructed from a genus-2 curve C in this way. We may therefore ask for an explicit construction of such a curve from a given Kummer quartic. This can indeed be done as follows.

Construction

(i) There are 16 tropes. We choose one of them, which we call D.

(ii) The intersection $Q \cap D$ is a double conic. Let I be the underlying reduced curve. Six of the singular points on Q are contained in I.

(iii) Take the double cover C of I ramified at these six points. This is a genus-2 curve.

Remarks 4.5. (i) This construction clearly yields a genus-2 curve C on the abelian surface A. The Albanese property of the Jacobian guarantees that A is at least isogenous to $J(C)$. They are actually isomorphic to each other.

(ii) If Q is defined over a base field k and D over an extension $k' \supseteq k$, then C is defined over k'. Indeed, the six ramification points form a $\mathrm{Gal}(\overline{k'}/k')$-invariant set. We will apply the construction only to the obvious tropes of the Kummer family, which are defined over the base field.

Fact 4.6. Let V' be an abstract Kummer surface over a finite field $\mathbb{F}_q$ and let V be its resolution of singularities. Then the $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-module $H^2_{\text{ét}}(V_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$ is reducible. A direct summand is isomorphic to $H^2_{\text{ét}}(A_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$ where A is the abelian surface covering V'. Its complement is described by the Galois operation on the 16 singular points.

Remark 4.7 (Frobenius eigenvalues for Kummer surfaces). In order to determine the eigenvalues of the Frobenius on $H^2_{\text{ét}}(V_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$, the usual method is to count the points on V defined over $\mathbb{F}_q$ and some of its extensions and to apply the Lefschetz trace formula [**24**, Chapter VI, Theorem 12.3].

For Kummer surfaces there is, however, a far better method. In fact, 16 eigenvalues are determined by the operation of Frobenius on the 16 singular points. Further, for A isogenous to the Jacobian $J(C)$, we have $H^2_{\text{ét}}(A_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l) \cong \Lambda^2 H^1_{\text{ét}}(C_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$. Thus, in order to determine the remaining six eigenvalues, it suffices to count the points on C. This is faster, as the problem is reduced to one dimension.

Proposition 4.8. *Let A be an abelian surface over an algebraically closed field. Suppose that* $\mathrm{End}(A)$ *is an order of a real quadratic number field. Then* $\mathrm{rk}\,\mathrm{NS}(A) = 2$.

Proof. According to [25, Section 21, Application III], one has $\mathrm{NS}(A)\otimes\mathbb{Q}\cong(\mathrm{End}(A)\otimes\mathbb{Q})^\dagger$, where $\dagger$ denotes the Rosati involution. As that is positive [25, Section 21, Theorem 1], it cannot be the conjugation on a real quadratic number field. Hence $\dagger=\mathrm{id}$, which implies the assertion. $\square$

REMARK 4.9. If the real multiplication is by an order in $\mathbb{Q}(\sqrt{d})$, then the discriminant of the Néron–Severi lattice is of square class $(-d)$. Indeed, [25, Section 21, Theorem 1] tells us that for H ample, $\Phi^*(H)\cdot H$ is a scalar multiple of $\mathrm{Tr}(\Phi^2)$. Working with $\Phi=1$ and $\Phi=1+\sqrt{d}$, we find the intersection matrix $\left(\begin{smallmatrix}2 & 2(1+d)\\ 2(1+d) & 2(1-d)^2\end{smallmatrix}\right)$ of determinant $(-16d)$.

PROPOSITION 4.10. *Let A be an abelian surface over $\mathbb{Q}$. Suppose that A has an endomorphism N defined only over a quadratic extension $F=\mathbb{Q}(\sqrt{D})$. Then, for every prime p that is inert in F and of good reduction, the following is true.*

If λ is an eigenvalue of Frob_p *on $H^1_{\text{ét}}(A_{\overline{\mathbb{F}}_p},\mathbb{Q}_l)$, then $(-\lambda)$ is an eigenvalue, too.*

Proof. The endomorphism N induces an endomorphism of $A_{\mathbb{F}_p}$, which we denote by $\underline{N}$. Clearly, $\underline{N}$ is defined over $\mathbb{F}_{p^2}$ but not over $\mathbb{F}_p$. This means that in the endomorphism ring R_p of $A_{\overline{\mathbb{F}}_p}$ we have $\mathrm{Frob}_{p^2}^{-1}\,\underline{N}\,\mathrm{Frob}_{p^2}=\underline{N}$, but the analogous statement is not true for Frob_p.

Thus, under the operation of Frob_{p^2} on R_p by conjugation, $\underline{N}$ lies in the $(+1)$-eigenspace. For the corresponding operation of Frob_p, this space decomposes into a $(+1)$-eigenspace and a (-1)-eigenspace. The latter is nonzero as $\underline{N}$ is not fixed under conjugation by Frob_p. Hence, there is some $J\in R_p$ anticommuting with Frob_p. This implies the assertion. $\square$

COROLLARY 4.11. *Let V be a Kummer surface over $\mathbb{Q}$ covered by the abelian surface A. Suppose that A has an endomorphism N defined only over a quadratic extension $F=\mathbb{Q}(\sqrt{D})$.*

Then, for every prime p that is inert in F and of good reduction, $\mathrm{rk}\,\mathrm{Pic}(V_{\overline{\mathbb{F}}_p})\geqslant 20$.

Proof. Recall that Kummer surfaces are elliptic [2, Chapter IX, Exercise 6]. Hence, the Tate conjecture is true for $V_{\overline{\mathbb{F}}_p}$.

Under the assumptions made, it is possible that the Frobenius eigenvalues on $H^1_{\text{ét}}(A_{\overline{\mathbb{F}}_p},\mathbb{Q}_l)$ are $\pm\sqrt{p}$ and $\pm i\sqrt{p}$. This yields Picard rank 22 over $\overline{\mathbb{F}}_p$. Except for this case, the Frobenius eigenvalues must be $\pm\lambda$ and $\pm\overline{\lambda}$ for a suitable $\lambda\in\mathbb{C}$. On $H^2_{\text{ét}}(A_{\overline{\mathbb{F}}_p},\mathbb{Q}_l)$, this leads to the eigenvalues p and $(-p)$, both with multiplicity two, as well as $(-\lambda^2)$ and $(-\overline{\lambda}^2)$. The Picard rank is at least 20. $\square$

5. *The tetrahedroid*

The tetrahedroid is another family of quartic surfaces that was studied in the 19th century. It was first considered by A. Cayley in [3].

LEMMA 5.1. *A family of quartics in $\mathbb{P}^3$ such that every member has exactly 16 A_1-singularities and no others is given by the equation*

$$\det\begin{pmatrix}0 & x_0^2 & x_1^2 & x_2^2 & x_3^2\\ x_0^2 & 0 & a_{01}^2 & a_{02}^2 & a_{03}^2\\ x_1^2 & a_{01}^2 & 0 & a_{12}^2 & a_{13}^2\\ x_2^2 & a_{02}^2 & a_{12}^2 & 0 & a_{23}^2\\ x_3^2 & a_{03}^2 & a_{13}^2 & a_{23}^2 & 0\end{pmatrix}=0$$

for parameters $a_{01},a_{02},a_{03},a_{12},a_{13},a_{23}\neq 0$.

REMARKS 5.2. (i) The equation of the tetrahedroid appears in this form in [4, p. 286].

(ii) We will write $T_{[a_{01},a_{02},a_{03},a_{12},a_{13},a_{23}]}$ for the quartic corresponding to the particular coefficient vector $(a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23})$.

(iii) Let the group $\mathbb{G}_m^4$ operate on the parameters according to the rule

$$(i, j, k, l)[a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23}] := [ija_{01}, ika_{02}, ila_{03}, jka_{12}, jla_{13}, kla_{23}].$$

Then, the quartics defined by a whole orbit are all isomorphic to each other, as one can see by left- and right-multiplying the matrix above by $\mathrm{diag}(1, i^2, j^2, k^2, l^2)$. Consequently, the tetrahedroid defines only a two-dimensional family in the moduli stack of all $K3$ surfaces. Actually, it is a subfamily of the Kummer quartics; see [4] and [14, Section 56].

REMARKS 5.3. (a) The sixteen singularities are $(0 : \pm a_{01} : \pm a_{02} : \pm a_{03})$, $(\pm a_{01} : 0 : \pm a_{12} : \pm a_{13})$, $(\pm a_{02} : \pm a_{12} : 0 : \pm a_{23})$, and $(\pm a_{03} : \pm a_{13} : \pm a_{23} : 0)$.

(b) The four planes, given by $\pm a_{23}x_1 \pm a_{13}x_2 \pm a_{12}x_3 = 0$, clearly contain six singular points each. For example, $((\pm a_{01}) : 0 : a_{12} : (-a_{13}))$, $((\pm a_{02}) : a_{12} : 0 : (-a_{23}))$, and $((\pm a_{03}) : a_{13} : (-a_{23}) : 0)$ satisfy the equation $a_{23}x_1 + a_{13}x_2 + a_{12}x_3 = 0$. Twelve more tropes are obtained in an analogous manner by distinguishing the first, second, or third coordinate instead of the zeroth one.

(c) Besides the tropes, there are four other particular planes related to this family of quartics. Actually, the coordinate planes contain exactly four singularities each. As these form a tetrahedron, they gave this family its name. There are no planes containing exactly four singular points on a general Kummer quartic.

PROPOSITION 5.4. *Let E_1 and E_2 be two elliptic curves. Fix an isomorphism of groups $\phi\colon E_1[2] \to E_2[2]$ and let*

$$A := (E_1 \times E_2)/\langle (x, \phi(x)) \mid x \in E_1[2] \rangle$$

be the corresponding abelian surface, covered four-to-one by $E_1 \times E_2$.

Then, the Kummer surface corresponding to A is given by a tetrahedroid.

Proof. We describe the elliptic curves as intersections of two quadrics in $\mathbb{P}^3$,

$$\begin{aligned} E_1\colon\ & x_1^2 = x_0^2 - x_2^2, \quad x_3^2 = x_0^2 - \kappa_1 x_2^2;\\ E_2\colon\ & y_1^2 = y_0^2 - y_2^2, \quad y_3^2 = y_0^2 - \kappa_2 y_2^2. \end{aligned}$$

We have

$$j(E_1) = 256(\kappa_1^2 - \kappa_1 + 1)^3/\kappa_1^2(\kappa_1 - 1)^2$$

and the analogous formula for E_2. Thus, these equations define general families of elliptic curves. The morphism

$$E_1 \times E_2 \longrightarrow \mathbb{P}^3, \quad ((x_0 : x_1 : x_2 : x_3), (y_0 : y_1 : y_2 : y_3)) \mapsto (x_2y_3 : x_1y_1 : x_3y_2 : x_0y_0)$$

is generically eight-to-one onto the tetrahedroid $T_{[\sqrt{\kappa_2 - 1}, 1, \sqrt{\kappa_2}, \sqrt{\kappa_1 - 1}, i, \sqrt{\kappa_1}]}$. It factors through A and even through the Kummer surface associated with it. □

REMARKS 5.5. (i) It is not hard to see that every tetrahedroid can be obtained from two elliptic curves in this way.

(ii) From the point of view of the present article, Proposition 5.4 is a purely algebraic statement. We even checked the assertions on the morphism using `Magma`. It was, however, originally discovered by H. Weber [31, p. 353] in the guise of a parametrization of the tetrahedroid by elliptic functions; see [14, Chapter XVIII].

Proposition 5.6 (Kummer quartics with two equal coefficients). *Let $V := V_{[a,a,c]}$ be the Kummer quartic for the coefficients $[a, a, c]$. Then, V is linearly isomorphic to the tetrahedroid*

$$T_{\left[\sqrt{c+1},\sqrt{c-1},X\sqrt{c-1},X\sqrt{c-1},\sqrt{c-1},2X(X+a)\sqrt{c-1}\right]}.$$

Here, X is a solution of the equation $X^2 + 2aX + 1 = 0$.

Proof. The isomorphism from the tetrahedroid to V is given explicitly by the linear map $\mathbb{P}^3 \to \mathbb{P}^3$,

$$\begin{aligned}&(t_1 : t_2 : t_3 : t_4)\\ &\quad\mapsto \left(\left(-t_2 - \frac{t_3 - Xt_4}{1 - X^2}\right) : \left(-t_1 + \frac{t_4 - Xt_3}{1 - X^2}\right) : \left(t_2 - \frac{t_3 - Xt_4}{1 - X^2}\right) : \left(t_1 + \frac{t_4 - Xt_3}{1 - X^2}\right)\right). \quad \square\end{aligned}$$

Remark 5.7. One might be interested in determining the two elliptic curves E_1 and E_2 which correspond to $V_{[a,a,c]}$, that is, which satisfy $(E_1 \times E_2)/\langle(x, \phi(x))\rangle \cong V_{[a,a,c]}$. This leads to a simple calculation, but the explicit formulas become rather lengthy. Interestingly, the two j-invariants are defined in the quadratic field extension $\mathbb{Q}(a, c)(\sqrt{4a^2 - 2c - 2})$ and are conjugate to each other. Their trace is

$$\begin{aligned}&(1024a^{10}c^2 + 2048a^{10}c + 1024a^{10} - 512a^8c^3 - 4608a^8c^2 - 7680a^8c - 3584a^8\\ &+ 32a^6c^4 + 1568a^6c^3 + 7776a^6c^2 + 10976a^6c + 4736a^6 - 72a^4c^4 - 1680a^4c^3\\ &- 6016a^4c^2 - 7280a^4c - 2872a^4 + 54a^2c^4 + 702a^2c^3 + 2010a^2c^2 + 2106a^2c\\ &+ 760a^2 - \tfrac{27}{2}c^4 - 81c^3 - 180c^2 - 175c - \tfrac{125}{2})/(a - 1)(a + 1)(b - 1)^2(b + 1)^2,\end{aligned}$$

while their norm turns out to be

$$\frac{(16a^4b^2 + 48a^4 - 24a^2b^2 - 32a^2b - 72a^2 + 9b^2 + 30b + 25)^3}{16(a - 1)^2(a + 1)^2(b - 1)^4(b + 1)^2}.$$

Remarks 5.8. (i) The case of three equal coefficients is even more special. In some sense, the quartics $V_{[a,a,a]}$ are tetrahedroids in three distinct ways.

It turns out that, in this situation, the resulting elliptic curves are related by an isogeny of order three. In fact, it is easy to check that the resulting pair of j-invariants is a zero of the third classical modular polynomial.

Consequently, the Picard rank of a Kummer surface with three equal coefficients is at least 19. The additional divisor leading to a Picard rank higher than 18 is the image of the graph of the 3-isogeny under the two-to-one covering described in Proposition 5.4.

(ii) There is another case which is special. Consider the quartics $V_{[0,0,c]}$. Then, the j-invariants of the corresponding elliptic curves are defined in $\mathbb{Q}(c)$ and equal to each other. We have

$$j(E_1) = j(E_2) = \frac{1728c^3 + 8640c^2 + 14400c + 8000}{c^3 - c^2 - c + 1}.$$

Consequently, the Picard rank of a Kummer surface with two coefficients being zero is at least 19. The additional divisor leading to a Picard rank higher than 18 is the image of the diagonal.

6. *Experiments: the Picard ranks over $\overline{\mathbb{Q}}$*

A sample of Kummer surfaces. We inspected the Kummer surfaces $Q_{[a,b,c]}$ given by the Kummer coefficients $a, b, c = -30, \ldots, 30$. Owing to symmetry, we could restrict our attention to the case where $|a| \leqslant b \leqslant c$. Recall that one can always change the signs of two coefficients simultaneously. Hence $b, c \geqslant 0$ was assumed. The coefficient vectors $[3, 3, 17]$,

$[2,2,7]$ and $[2,7,26]$, as well as those containing ± 1, were excluded from the sample as the corresponding surfaces have singularities of types worse than A_1.

6.1. For each surface Q in the sample, we first determined, using Construction 4.4, the genus-2 curve C such that V is the Kummer surface corresponding to $J(C)$. Then, for every prime number below 1000, we counted the numbers of points on C over $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$. From these data, we computed the characteristic polynomial of the Frobenius on the l-adic cohomology of the resolution V.

From the characteristic polynomial, we read off the rank of $\mathrm{Pic}(V_{\overline{\mathbb{F}}_p})$ and, using the Artin–Tate formula in Conjecture 1.12, computed the square class of the discriminant. Note that the Artin–Tate formula is applicable, since every Kummer surface is elliptic.

The Picard ranks over $\overline{\mathbb{Q}}$. A generic Kummer surface is of geometric Picard rank 17. In the case of two Kummer coefficients having the same absolute value, Proposition 5.6, together with Remark 2.5(ii), shows that the surface is a tetrahedroid. Then, the Picard rank is at least 18. Thus, we distinguished between these two cases. The possibilities where all three coefficients coincide, at least up to symmetry, or where two coefficients vanish were treated as being somehow exceptional.

Being a bit sloppy at the beginning, in the first case we tested whether an upper bound of 17 is provable by van Luijk's method; in the second case, we awaited an upper bound of 18. Table 1 shows the distribution of the biggest prime that had to be considered in order to prove the expectation.

The 18 examples left. Let us take a closer look at the Kummer quartics left.

Examples 6.2. Among the Kummer quartics whose coefficients had three distinct absolute values, twelve examples remained. For these, only a rank bound of 18 could be established. Using `Magma`, we calculated the corresponding genus-2 curves C_i and determined their periods at high precision.

(i) Consider the Kummer quartics for the coefficient vectors $[2,3,13]$, $[-3,4,19]$, $[-3,5,11]$, $[-2,7,23]$, $[-2,8,17]$, $[-2,9,14]$, and $[0,4,7]$.

Table 1. *Distribution of the biggest prime used for rank* 17 *(left) and rank* 18 *(right).*

Prime	No. cases finished	No. cases left
7	57	7656
11	287	7369
13	713	6656
17	1229	5427
19	1308	4119
23	1215	2904
29	1004	1900
31	759	1141
37	551	590
41	320	270
43	143	127
47	59	68
53	28	40
59	17	23
61	6	17
67	3	14
73	1	13
83	1	12

Prime	No. cases finished	No. cases left
5	156	1495
7	66	1429
11	193	1236
13	253	983
17	288	695
19	132	563
23	117	446
29	116	330
31	82	248
37	81	167
41	73	94
43	24	70
47	18	52
53	15	37
59	13	24
61	6	18
67	3	15
71	2	13
73	4	9
79	2	7
101	1	6

In these cases, it turned out that the Jacobians $J(C_i)$ are isogenous to products of two elliptic curves. Hence, the geometric Picard ranks are indeed equal to 18.
The isogenies are all of degree 16. Their kernels are groups of type $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. The j-invariants of the elliptic curves are conjugate to each other in quadratic number fields. We summarize them in Table 2.

(ii) Consider the Kummer quartics given by the coefficient vectors $[2, 7, 17]$, $[2, 9, 26]$, $[2, 17, 26]$, $[3, 9, 19]$, and $[0, 8, 15]$.
Our calculations showed that the corresponding abelian surfaces have real multiplication by orders in $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{5})$, and $\mathbb{Q}(\sqrt{5})$, respectively. This implies that the Picard ranks are equal to 18.

The nontrivial endomorphisms are expected to be defined over the quadratic number fields $\mathbb{Q}(\sqrt{30})$, $\mathbb{Q}(\sqrt{11})$, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-1})$, and $\mathbb{Q}(\sqrt{2})$. In fact, the primes leading to Picard rank 18 are all split for the corresponding field. Compare Corollary 4.11.

6.3. Consider the Kummer quartics for the coefficient vectors $[5, 5, 17]$, $[2, 2, 17]$, $[-4, 4, 9]$, $[-3, 7, 7]$, $[-2, 11, 11]$, and $[0, 5, 5]$. For these, the situation is as follows.

One finds rank 20 at several primes. Discriminants of various square classes appear such that a rank bound of 19 is established.

As two of the Kummer coefficients are equal, the corresponding abelian surfaces are isogenous to products of two elliptic curves. By specializing the calculation discussed in Remark 5.7, one can determine the corresponding j-invariants. It turns out in every case that the corresponding elliptic curves are isogenous to each other. Thus, we have Picard rank 19. The isogenies are of degrees 5, 5, 4, 4, 4, and 4, respectively.

Examples 6.4. Let us present two examples in detail.

(i) Let T_1 be the Kummer quartic for the coefficient vector $[5, 5, 17]$. We find rank 20 at $p = 5$, 7, 13, 17, 19, 23, 29 and several other primes. The rank bound 19 is proven, as many distinct square classes of discriminants occur.

Further, as two of the Kummer coefficients are equal, the corresponding abelian surface is isogenous to a product of two elliptic curves. By specializing the calculation discussed in

Table 2. *j-invariants of the corresponding elliptic curves.*

Vector	j_1, j_2
$[2, 3, 13]$	$\frac{8000}{21609}\left[38155 \pm 16152\sqrt{2}\,\right]$
$[-3, 4, 19]$	$\frac{64}{164025}\left[1082783 \pm 399784\sqrt{-2}\,\right]$
$[-3, 5, 11]$	$\frac{16}{5625}\left[17903 \pm 64596\sqrt{-1}\,\right]$
$[-2, 7, 23]$	$\frac{2000}{10673289}\left[-614135 \pm 4744012\sqrt{-2}\,\right]$
$[-2, 8, 17]$	$\frac{250}{21609}\left[-50045 \pm 45683\sqrt{-3}\,\right]$
$[-2, 9, 14]$	$\frac{16}{6426225}\left[327552721 \pm 229629540\sqrt{-1}\,\right]$
$[0, 4, 7]$	$\frac{16}{5625}\left[17903 \pm 64596\sqrt{-1}\,\right]$

Remark 5.7, one finds the j-invariants
$$j_1 = \frac{85184}{3} \quad \text{and} \quad j_2 = \frac{58591911104}{243}.$$
The pair (j_1, j_2) is a zero of the fifth modular polynomial. Hence, between the two elliptic curves, there is an isogeny of order five. We have Picard rank 19.

(ii) Let T_2 be the Kummer quartic for the coefficient vector $[2, 2, 17]$. For this surface, we find rank 20 at $p = 7$, 11, 13, 17, 23, 29 and several other primes. Many distinct square classes of discriminants appear. Hence, the rank bound 19 is proven.

Here, the two j-invariants are defined in $\mathbb{Q}(\sqrt{-5})$. They are the roots of the polynomial
$$X^2 + \frac{21180800}{243}X + \frac{1693669888000}{729}.$$
Again, the corresponding elliptic curves turn out to be 5-isogenous. This confirms Picard rank 19.

Expected rank 19. In the case of three equal coefficients or two coefficients equal to zero, we know that the Picard rank is at least 19. In 84 of the 88 surfaces, the reductions modulo p provided an upper bound of 19. The biggest prime that had to be used was 37.

The cases $[0, 0, 0]$, $[-5, 5, 5]$, $[-2, 2, 2]$, and $[7, 7, 7]$ remained. Here, the corresponding elliptic curves have complex multiplication. This shows that the corresponding Kummer surfaces indeed have geometric Picard rank 20.

Example 6.5. Consider, for instance, the case $[7, 7, 7]$. Then, the two j-invariants are the roots of the polynomial $X^2 - 37018076625X + 153173312762625$. The corresponding elliptic curves have complex multiplication by an order in $\mathbb{Q}(\sqrt{-15})$.

Testing isomorphy, I. As a byproduct of the computations, we tried to prove that the surfaces in our sample are pairwise non-isomorphic. For this, it would suffice to show that for each pair of surfaces, there exists a prime where both have good reduction, but the geometric Picard groups differ in rank or discriminant. Actually, the data for $p \leqslant 59$ contained enough information to do this, but there were 41 pairs of surfaces that could not be separated.

The point here is that the test actually tries to prove that the corresponding abelian surfaces are non-isogenous. But in these 41 cases, the surfaces are isogenous to each other. To be more precise, we found 17 pairs, four triples, and two quadruples of mutually isogenous abelian surfaces.

Example 6.6. The abelian surfaces corresponding to $V_{[2,2,9]}$ and $V_{[3,3,19]}$ are isogenous. Hence, the test described above has no chance of working.

In fact, $V_{[2,2,9]}$ is covered eight-to-one by $E_1 \times E_2$ while $V_{[3,3,19]}$ is covered eight-to-one by $E_3 \times E_4$, with $j(E_1)$ and $j(E_2)$ being the zeros of
$$X^2 + \frac{1114112}{25}X + \frac{589752696832}{225}$$
and $j(E_3)$ and $j(E_4)$ the zeros of
$$X^2 - \frac{281615072}{2025}X + \frac{15000601854041872}{164025}.$$
It is easy to check that E_1 and E_3, as well as E_2 and E_4, are connected by isogenies of order four. Hence $E_1 \times E_2$ and $E_3 \times E_4$ are 16-isogenous.

An isomorphism between the quotients as described in Proposition 5.4 would yield a 16-isogeny
$$E_1 \times E_2 \longrightarrow E_1 \times E_2/\langle (x, \phi(x)) \mid x \in E_1[2] \rangle \cong E_3 \times E_4/\langle (x, \phi'(x)) \mid x \in E_3[2] \rangle \longrightarrow E_3 \times E_4$$

as well. But in its kernel are the 2-torsion points $(x, \phi(x))$ for $x \in E_1[2]$, which are clearly not in the kernel of the direct product of two 4-isogenies. This shows that $V_{[2,2,9]}$ and $V_{[3,3,19]}$ are not isomorphic either.

Testing isomorphy, II. For each of the 41 pairs, we numerically calculated the periods of the corresponding abelian surfaces. From these, we determined a minimal isogeny. It turned out that the surfaces corresponding to the coefficient vectors $[-3, 7, 7]$ and $[0, 5, 5]$ were actually isomorphic to each other. This was, however, the only such case among the critical pairs.

Summary. We considered the resolutions of 9452 Kummer quartics with exactly 16 singularities of type A_1. It turned out that the upper bounds for the Picard ranks provided by the reductions modulo p were sharp in every case. However, for several examples, rather large primes up to $p = 101$ had to be considered. We had Picard rank 17 7701 times, Picard rank 18 1657 times, and Picard rank 19 90 times. Further, there were four surfaces of Picard rank 20 in the sample.

7. *Some more statistics*

Example 7.1 (All primes less than 10 000 for a typical surface). Let us take a closer look at a particular example. We selected the surface with Kummer coefficients [3, 11, 21], but many others would be representative as well.

There are only five primes $p \leqslant 10\,000$ such that the reduction modulo p of $V_{[3,11,21]}$ is not a quartic having 16 singular points of type A_1. They are 2, 3, 5, 11, and 17. In the range considered, 1224 primes lead to a reduction of Picard rank 18. Further, there are 69 primes leading to a reduction of rank 20. These seem to be rather evenly distributed within the range, the smallest one being 7 and the largest one 9677. Finally, there is the prime 4583, which leads to a reduction of Picard rank 22.

In the cases of reduction to Picard rank 18, we found 586 different square classes for the discriminant. As for many of the surfaces in our sample, the most frequently occurring square class was (-1). In the example selected, it appeared 376 times.

Discriminants: the special case of rank 17. In the special case of a rank-17 surface, we counted how many square classes of discriminants occurred when reducing to surfaces of Picard rank 18 modulo various primes. There are 168 prime numbers in our computational range ($p < 1000$). For a fixed surface, between 44 and 89 distinct square classes were found; these results are shown in Figure 1.

In total, we found 541 distinct square classes of discriminants. Some of them occurred only for one surface and one prime. On the other hand, the class of (-1) appeared 134 553 times. The surfaces with Kummer coefficients $[-3, 9, 17]$ and $[-3, 10, 29]$ both had the most repetitions for one square class. This was the class of (-1), which occurred 43 times.

The average value for a prime. For simplicity, let us restrict our attention to surfaces of Picard rank 17. For every prime number p, we counted how many of the surfaces in our sample had good reduction modulo p. We determined the proportion of those having reduction to rank higher than 18. The results are shown in Figure 2. According to this graph, the proportion is close to $C/\sqrt{p}$ for $C \approx 2$.

The average value for a surface. On the other hand, for every surface of Picard rank 18 or less in the sample, we counted how many primes below 1000 lead to a reduction of geometric Picard rank 18 over $\overline{\mathbb{F}}_p$. Let us visualize the result in a histogram, shown in Figure 3.

The histogram clearly suggests that there are two kinds of examples. For the first kind, the probability that the reduction has rank 18 is between 1/4 and 1/2. For the second kind, this probability is between 3/4 and 1.

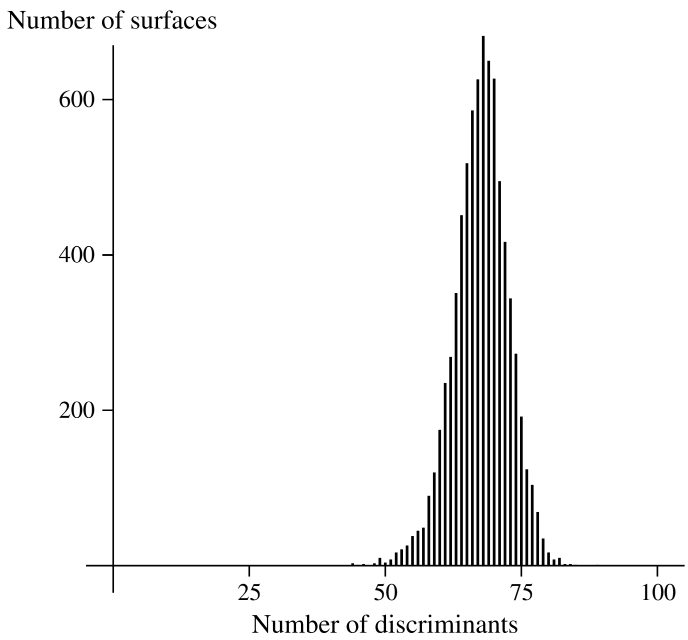

FIGURE 1. *Number of distinct square classes of discriminants at primes with reduction to rank* 18.

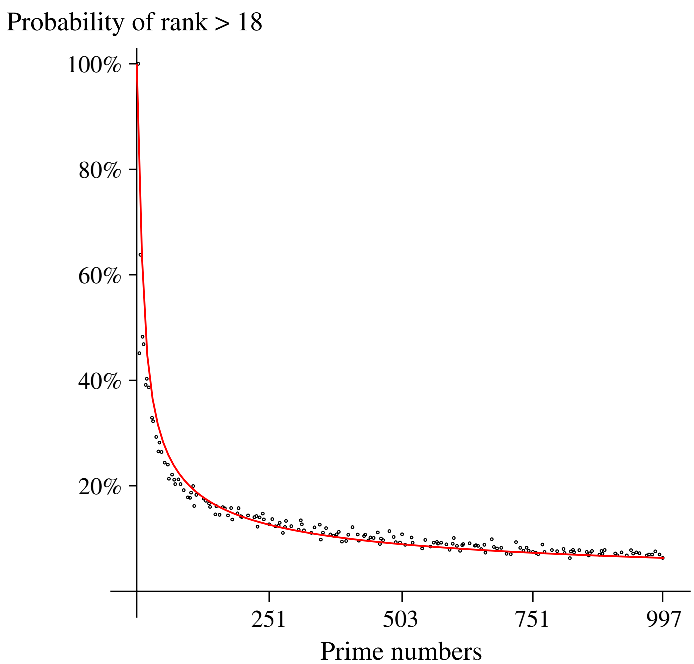

FIGURE 2. *Distribution of the proportion of the surfaces with reduction to rank higher than* 18.

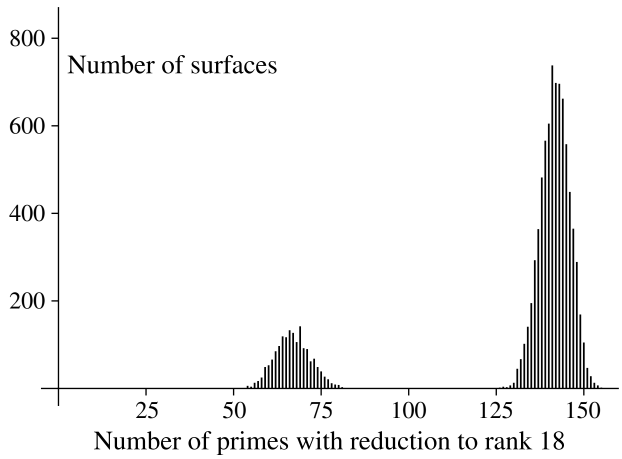

FIGURE 3. *Distribution of the number of the primes with reduction to rank* 18.

It turns out that most of the examples with two Kummer coefficients being equal (up to sign) belong to the first kind. The only examples in the first group that are not of this form are given by the coefficient vectors $[3,9,19]$, $[2,3,13]$, $[2,7,17]$, $[2,9,26]$, $[2,17,26]$, $[-3,4,19]$, $[-3,5,11]$, $[-2,7,23]$, $[-2,8,17]$, $[-2,9,14]$, $[0,4,7]$, and $[0,8,15]$. Further, there are some examples with two equal coefficients that belong to the second group. These are $[3,3,9]$, $[3,3,15]$, $[4,4,13]$, $[4,4,23]$, $[4,4,29]$, $[6,6,21]$, $[7,7,25]$, $[8,8,29]$, $[2,2,5]$, $[-6,6,27]$, $[-5,5,23]$, $[-4,4,19]$, $[-3,3,15]$, $[-2,2,11]$, and $[-2,2,25]$.

An explanation. For the tetrahedroid case, an explanation is given by the following fact.

FACT 7.2. Let $V_{[a,a,c]}$ be a Kummer surface with two equal coefficients. Suppose that $4a^2-2c-2$ is not a perfect square.

Then, for every prime p that is inert in $F=\mathbb{Q}(\sqrt{4a^2-2c-2})$ and of good reduction, $\operatorname{rk}\operatorname{Pic}(V_{\overline{\mathbb{F}}_p})\geqslant 20$.

Proof. The corresponding abelian surface is isogenous to the product of two elliptic curves. As noted in Remark 5.7, the j-invariants are two elements conjugate in F. Reducing the surface modulo a prime inert in F leads to two elliptic curves that are isogenous via the Frobenius endomorphism. This shows that all inert primes yield an upper bound of at least 20 for the geometric Picard rank. $\square$

QUESTIONS 7.3. (i) For a surface V, put $N_V(B):=\#\{p\in P_V\mid p\leqslant B\}$ where

$$P_V:=\#\{p \text{ prime}\mid \operatorname{rk}\operatorname{Pic}(V_{\overline{\mathbb{F}}_p})>18 \text{ or } V \text{ has bad reduction at } p\}.$$

Is there a monotonically decreasing function h_V such that

$$N_V(B)\sim\int_B^2\frac{h_V(t)}{\log t}\,dt\,?$$

Can h_V be given explicitly?

(ii) Suppose that $\operatorname{rk}\operatorname{Pic}(V_{\overline{\mathbb{Q}}})=17$. Does h_V then converge to 0 for $t\to\infty$? The graph in Figure 2 might suggest that $h_V(t)=C_V/\sqrt{t}$ for a constant C_V. Is h_V perhaps independent of V?

(iii) For a fixed Kummer surface of geometric Picard rank 17, are there infinitely many primes with reduction to rank 18? Are there infinitely many primes with reduction to rank higher than 18?

REMARK 7.4. In relation to these questions, the reader may want to consult [**26**], for instance Conjecture 5.1 formulated there.

REMARK 7.5. When $\operatorname{rk}\operatorname{Pic}(V_{\overline{\mathbb{Q}}}) = 18$, the situation is typically different. For example, when two Kummer coefficients are equal, we saw in Fact 7.2 that $P_{V_{[a,a,c]}}$ has density at least $1/2$ unless $4a^2 - 2c - 2$ is a perfect square. According to Proposition 4.10, the same is true when the abelian surface corresponding to V has real multiplication by an endomorphism defined over a proper field extension of $\mathbb{Q}$. Note that the latter case actually subsumes the former, as the abelian variety corresponding to $V_{[a,a,c]}$ is isogenous to the product of two elliptic curves and therefore has real multiplication.

8. *Our data*

8.1. The raw data from the experiments are available on both authors' webpages.

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