A Note on Dirac Fields with Vanishing Energy-Momentum and Spin Tensor

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Abstract

We characterize Dirac fields in a gravitational background with torsion, for which the energy-momentum tensor and the spin tensor vanish.

Einstein's field equations admit exact solutions with a Dirac field ψ coupled to the gravitational field in such a way that the (minimally coupled) energy-momentum tensor of the spinor field vanishes (see Ref. 1 and the references cited there). This means that the same geometry is compatible with $\psi = 0$ as well as with a special non-vanishing matter field, thus demonstrating a certain insensitivity of the geometry concerning the matter distribution. Exact solutions of this type are sometimes called "ghost solutions". In the Einstein-Cartan and more generally the "Poincaré gauge theory" (see Ref. 2 for reviews), matter fields couple to the geometry not only through the (canonical) energy-momentum tensor, but also via the spin tensor. In the following we address the question whether Dirac fields exist for which both, the energy-momentum and the spin tensor, vanish. The following results are extensions of our previous work [1] from which we take the notation. They only rely on the kinematics of space-times with torsion, but not on a particular choice of field equations for metric and torsion.

Lemma 1. Let ψ be a Dirac spinor field on a four-dimensional space-time M. If the spin tensor vanishes, i.e. $\bar{\psi}\gamma^5\gamma^k\psi = 0$, then

$$\bar{\psi}\psi = 0 = \bar{\psi}\gamma^5\psi \; .$$

Proof: $(\bar{\psi}\psi)^2 + (\bar{\psi}\gamma_5\psi)^2 > 0$ would contradict our assumption of a vanishing spin tensor by use of part (1) of the theorem in Ref. 1, Appendix A. ■

Lemma 2. Let ψ be a Dirac spinor field on a four-dimensional space-time M. The following statements are then equivalent:
(1) The spin tensor vanishes, i.e. $\bar{\psi}\gamma^5\gamma^k\psi = 0$.
(2) There is (locally) an orthonormal frame field with respect to which the spinor field has the form

$$\psi = \mu \begin{pmatrix} 1 \\ 0 \\ 0 \\ e^{i\phi} \end{pmatrix}$$

with real functions μ and ϕ.
Proof: A direct calculation shows that (2) implies (1). Let us now assume that (1) holds. Then Lemma 1 and part (2) of the theorem in Ref. 1, Appendix A, ensure the local existence of an orthonormal frame field with respect to which

$$\psi = \mu \begin{pmatrix} 1 \\ 0 \\ if_3 \\ -f_2 + if_1 \end{pmatrix}$$

with real functions μ and f_α, $\alpha = 1, 2, 3$, restricted by $\sum_\alpha (f_\alpha)^2 = 1$. The condition of vanishing spin tensor now reduces this form of the spinor to the one given in (2). ■

Theorem. The following statements are equivalent for a Dirac spinor field on a four-dimensional space-time:
(1) The spin tensor and the canonical energy-momentum tensor vanish.
(2) There is (locally) an orthonormal frame field with respect to which the spinor field has the form

$$\psi = \mu \begin{pmatrix} 1 \\ 0 \\ 0 \\ e^{i\phi_0} \end{pmatrix}$$

with a real function μ and a real constant ϕ_0.
Proof: Since

$$\begin{aligned} \bar{\psi}\gamma_j D\psi &= \bar{\psi}\gamma_j d\psi + \frac{1}{8}\omega^{kl}\,\bar{\psi}\gamma_j[\gamma_k, \gamma_l]\psi \\ &= \bar{\psi}\gamma_j d\psi + \frac{1}{2}\omega^{jl}\,\bar{\psi}\gamma^l\psi - \frac{1}{4}\epsilon_{jmkl}\omega^{kl}\,\bar{\psi}\gamma^m\gamma_5\psi \end{aligned}$$

and $\mathrm{Re}(\bar{\psi}\gamma^l\psi) = 0$, the vanishing of the spin tensor and the energy-momentum tensor $T_{ij} = -\mathrm{Re}(\bar{\psi}\gamma_j\nabla_i\psi)$ implies

$$\mathrm{Re}(\bar{\psi}\gamma_j d\psi) = 0$$

with respect to an arbitrary orthonormal frame field. Inserting the form of the spinor field given in the preceding Lemma with respect to a special orthonormal frame field, the last condition amounts to $\phi =$ constant. Thus (1) implies (2). Furthermore, it can be checked directly that (2) implies (1). ■

For Dirac spinor fields described by the above theorem the system of Einstein-Cartan or more generally Poincaré gauge field equations reduces to the vacuum gravitational field equations and the linear Dirac equation. In the orthonormal frame field to which condition (2) of the theorem refers, we have

$$(j_k) := i(\bar{\psi}\gamma_k\psi) = 2\mu^2 \begin{pmatrix} -1 \\ -\sin(\phi_0) \\ \cos(\phi_0) \\ 0 \end{pmatrix}$$

which is a null vector: $j_k j^k = 0$.

Remark. The spinor field determined by the previous theorem satisfies the modified Majorana condition

$$\bar{\psi} = \psi^T C \quad \text{with} \quad C = -ie^{-i\phi_0}\gamma^2 \ .$$

It has been observed in Ref. 3 that, for a spinor which is constant with respect to some orthonormal frame field, the energy-momentum and spin tensor vanish. ■

The Dirac equation

$$([\partial_j + \frac{1}{2}\,{}_V\omega_j - \frac{1}{2}\,Q_j]\gamma^j + \frac{1}{2}\,{}_A\omega_j\,\gamma^j\gamma^5 + m)\psi = 0$$

evaluated with the form of the spinor field given in the above theorem reads

$$\begin{aligned} (A_1 - iA_2)\,e^{i\phi_0} - iA_0 - \frac{1}{2}\,{}_A\omega_3 + m &= 0 \\ (A_3 + \frac{i}{2}\,{}_A\omega_0)\,e^{i\phi_0} + \frac{1}{2}({}_A\omega_1 + i\,{}_A\omega_2) &= 0 \end{aligned}$$

where

$$A_j := \partial_j \ln|\mu| + \frac{1}{2}\,{}_V\omega_j - \frac{1}{2}\,Q_j \ .$$

Splitting these equations into real and imaginary parts, we find

$$\begin{aligned} {}_A\omega_1 &= {}_A\omega_0\,\sin(\phi_0) - 2A_3\,\cos(\phi_0) \\ {}_A\omega_2 &= -{}_A\omega_0\,\cos(\phi_0) - 2A_3\,\sin(\phi_0) \\ {}_A\omega_3 &= 2[m + A_1\cos(\phi_0) + A_2\,\sin(\phi_0)] \end{aligned}$$

and

$$A_0 = A_1\,\sin(\phi_0) - A_2\,\cos(\phi_0) \ .$$

The first three equations can be read as definitions of the components ${}_A\omega_\alpha$, $\alpha = 1,2,3$, of the connection, the last equation as definition of ${}_V\omega_0$. Hence we have solved the Dirac equation algebraically.

References

[1] A. Dimakis and F. Müller-Hoissen, "Solutions of the Einstein-Cartan-Dirac equations with vanishing energy momentum tensor", *J. Math. Phys.* **26** (1985) 1040

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[3] M. Seitz, "A solution of the Einstein-Dirac equations with an axially symmetric spin-1/2 ghost field", University of Cologne preprint (1988)