Deformation of Maps between Euclidean Spaces

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References:

Calc. Var. PDE 55:104 (2016)
arXiv:1608.05394

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Motivation

$f: M \to N$ smooth map between Riemannian manifolds (M, g_M) and (N, g_N).

Question:

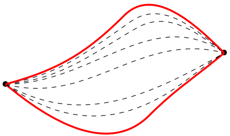
Is there a minimal representative in some equivalence class $[f]$?

1. What is a minimal representative?
2. What is the notion of equivalence in $[f]$?

Answer to 1: Two slides ahead!
Answer to 2: Equivalence with respect to homotopy.

Homotopy of maps

Two maps $g, h: X \to Y$ are homotopic iff there is $H \in C^0(X \times [0,1], Y)$ satisfying $H(x, 0) = g(x)$ and $H(x, 1) = h(x)$



$\rightsquigarrow$ How to deform the map?

How to deform a map? – The Setup

Manifolds: $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$
Smooth embedding: $F := \mathrm{id}_{\mathbb{R}^m} \times f : \mathbb{R}^m \hookrightarrow \mathbb{R}^m \times \mathbb{R}^n$

Graph of f

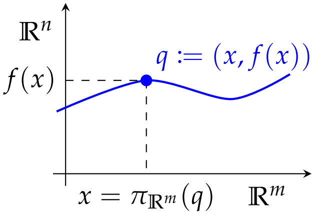
$$\Gamma(f) := F(\mathbb{R}^m) := \left\{(x, f(x)) : x \in \mathbb{R}^m\right\} \quad \subset \mathbb{R}^m \times \mathbb{R}^n$$

Projections

$$\pi_{\mathbb{R}^m} : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m, \quad \pi_{\mathbb{R}^m}(x, y) := x$$
$$\pi_{\mathbb{R}^n} : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n, \quad \pi_{\mathbb{R}^n}(x, y) := y$$

Metrics

$$\langle \cdot, \cdot \rangle = \pi^*_{\mathbb{R}^m} g_{\mathbb{R}^m} + \pi^*_{\mathbb{R}^n} g_{\mathbb{R}^n}$$
$$g := F^* \langle \cdot, \cdot \rangle$$



How to deform a map? – The Setup

Induced metric on $\mathbb{R}^m$: $\mathrm{g} = \mathrm{g}_{\mathbb{R}^m} + f^*\mathrm{g}_{\mathbb{R}^n}$

Levi-Civita Connections: (flat) D on $\mathbb{R}^m \times \mathbb{R}^n$ $\quad \nabla := \nabla^{\mathrm{g}}$ on $\mathbb{R}^m$

Second Fundamental Tensor (*extrinsic curvature*)

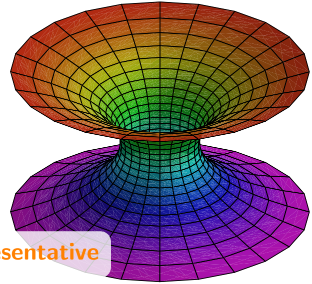
$$A \in \Gamma\left(\mathrm{Sym}(T^*\mathbb{R}^m \otimes T^*\mathbb{R}^m) \otimes F^*T(\mathbb{R}^m \times \mathbb{R}^n)\right),$$
$$A(u,v) := \mathrm{D}_{\mathrm{d}F(u)}\,\mathrm{d}F(v) - \mathrm{d}F(\nabla_u v)$$

Mean curvature vector:

$\{e_1, \ldots, e_m\}$ local g-orthonormal frame. Then

$$\vec{H} := \sum_{k=1}^{m} A(e_k, e_k) \qquad \in \Gamma(T^\perp \mathbb{R}^m)$$

Minimal map: $\vec{H} = 0$ **Minimal representative**



How to deform a map? – Mean Curvature Flow

Time-dependent family of immersions:

$$F : \mathbb{R}^m \times [0, T) \to \mathbb{R}^m \times \mathbb{R}^n, \qquad F_t(x) := F(x, t)$$

for some $T > 0$.

$F_t(\mathbb{R}^m)$ **evolves under mean curvature flow**, if

$$\begin{cases} \partial_t F(x,t) = \vec{H}(x,t) & \forall (x,t) \in \mathbb{R}^m \times [0,T) \\ F_0(x) = (x, f(x)) \end{cases}$$

(Some) Questions:

1. Short-time existence?
2. Regularity / singularities?

} ($\mathbb{R}^m$ is non-compact)

3. Does $F_t(M)$ stay graphic? I.e. is $F_t \circ \varphi_t(x) = (x, f_t(x))$?
4. Long-time existence? Convergence to minimal representative?
5. . . .

The Mean Curvature Flow Equation

In local coordinates:

$$\frac{\partial F}{\partial t} = \vec{H} = \sum_{k=1}^{m} A(e_k, e_k) = \sum_{i,j=1}^{m} \mathrm{g}^{ij}(\nabla\, \mathrm{d}F)(\partial_i, \partial_j) = \Delta_{\mathrm{g}} F$$

Observations:

- Equation is invariant under tangential diffeomorphisms
- g^{ij} = inverse metric, depends on F and $\mathrm{d}F$

Type of Equation

The mean curvature flow equation is a (degenerate) quasilinear parabolic PDE

Short Time Existence

Known for *compact* manifolds.

Short-Time Existence

MCF system: $\partial_t F = \vec{H}\,, \qquad F(x) = (x, f_0(x))$

Fix parametrization, i. e. $F \circ \varphi_t(x) = (x, f_t(x))$:

$$\frac{\partial f}{\partial t}(x,t) = \sum_{i,j=1}^{m} \tilde{g}^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j}\,, \qquad f(x,0) = f_0(x) \qquad \text{(pMCF)}$$

$f = (f^\alpha)_{\alpha=1}^n$ = height functions of the graph

$\tilde{g}^{ij}$ = inverse of $\tilde{g}_{ij} := \delta_{ij} + g_{\mathbb{R}^n}(\partial_i f, \partial_j f)$.

[parametric mean curvature flow]

Theorem (Chau, Chen, He, 2012)

Suppose $f_0 : \mathbb{R}^m \to \mathbb{R}^n$ is a smooth function such that $\sup_{\mathbb{R}^m} \|D^l f_0\| < \infty$ for each $l \geq 1$. Then (pMCF) has a short-time smooth solution $f(x,t)$ with initial condition f_0, such that $\sup_{x \in \mathbb{R}^m} \|D^l f(x,t)\| < \infty$ for each $l \geq 1$ and t.

Solutions with Bounded Geometry

Definition

$f:\mathbb{R}^m\times[0,T)\to\mathbb{R}^n$ has **bounded geometry**, if $\sup_{x\in\mathbb{R}^m}\|\mathrm{D}^l f(x,t)\|<\infty$ for each $l\geq 1$ and $t\in[0,T)$.

Definition

$F:\mathbb{R}^m\times[0,T)\to\mathbb{R}^m\times\mathbb{R}^n$ has **bounded geometry**, if $\forall t\in[0,T)$ and $k=0,1,2,\dots$

$$\sup_{x\in\mathbb{R}^m}\|\nabla^k A(x,t)\|<\infty\,,\qquad C_1(t)\mathrm{g}_{\mathbb{R}^m}\leq\mathrm{g}\leq C_2(t)\mathrm{g}_{\mathbb{R}^m}$$

with $0<C_1(t)\leq C_2(t)<\infty$.

[Def's equivalent for $F_t\circ\varphi_t(x)=(x,f_t(x))$]

Note: For $t\to T$ ...

- ... $C_1(t)$ may approach 0,
- ... $C_2(t)$ may become unbounded,
- ... $\sup_{x\in\mathbb{R}^m}\|\nabla^k A(x,t)\|$ may become unbounded.

Length-Decreasing Maps

Definition (Length-decreasing maps)

A map $f : \mathbb{R}^m \to \mathbb{R}^n$ is strictly length-decreasing iff

$$\| df(v)\|_{\mathbb{R}^n} \leq (1-\delta)\|v\|_{\mathbb{R}^m}$$

for some (fixed) $\delta \in (0,1]$ and all $v \in \Gamma(T\mathbb{R}^m)$.

Example 1

- $f(x) := Ax + b, \qquad A \in \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n),\ b \in \mathbb{R}^n$
- $A^t A$ symmetric $\rightsquigarrow$ $A^t A$ is diagonalizable
- if $\sigma(A^t A) \in [0, 1-\delta]$, f is length-decreasing

Example 2

- $f : \mathbb{R}^m \to \mathbb{R}^n$ with $\| df(v)\|^2_{\mathrm{g}_{\mathbb{R}^n}} \leq c(1-\delta)\|v\|^2_{\mathrm{g}_{\mathbb{R}^m}}$ for $c > 0$
- Define $\widetilde{\mathrm{g}}_{\mathbb{R}^m} := c\mathrm{g}_{\mathbb{R}^m}$ and $\widetilde{\mathrm{g}}_{\mathbb{R}^n} := c^{-1}\mathrm{g}_{\mathbb{R}^n}$
- f is strictly length-decreasing w.r.t. $\widetilde{\mathrm{g}}_{\mathbb{R}^m}$ and $\mathrm{g}_{\mathbb{R}^n}$
- f is strictly length-decreasing w.r.t. $\mathrm{g}_{\mathbb{R}^m}$ and $\widetilde{\mathrm{g}}_{\mathbb{R}^n}$

Theorem (FL, Calc. Var. PDE 55:104 (2016))

Suppose $f_0 : \mathbb{R}^m \to \mathbb{R}^n$ is a Lipschitz continuous function and that there exists a fixed $\delta \in (0,1]$, such that

$$\| \mathrm{d}f_0(v)\|_{g_{\mathbb{R}^n}} \leq (1-\delta)\|v\|_{g_{\mathbb{R}^m}}$$

a.e. on $\mathbb{R}^m$ and for all $v \in \Gamma(T\mathbb{R}^m)$. Then MCF has a long-time smooth solution for all $t > 0$ with initial condition $F_0(x) := (x, f_0(x))$, such that the following statements hold.

1. *Along the flow, the evolving submanifold stays the graph of a strictly length-decreasing map f_t.*
2. *The mean curvature vector of the graph satisfies the estimate $t\|\vec{H}\|^2 \leq C$.*
3. *The spatial derivatives satisfy*

$$t^{k-1} \sup_{x\in\mathbb{R}^m} \|\mathrm{D}^k f_t(x)\|^2 \leq C_{k,\delta} \qquad \text{for all} \quad k \geq 2\,.$$

Moreover, $\sup_{x\in\mathbb{R}^m} \|f_t(x)\|^2 \leq \sup_{x\in\mathbb{R}^m} \|f_0(x)\|^2$ for all $t > 0$.

If in addition f_0 satisfies $f_0 \xrightarrow{\|x\|\to\infty} 0$, then $\|f_t(x)\| \to 0$ smoothly on compact sets of $\mathbb{R}^m$ for $t \to \infty$.

Length-decreasing property is preserved

Recall: $f : \mathbb{R}^m \to \mathbb{R}^n$ (strictly) length-decreasing
$\Leftrightarrow \; f^* g_{\mathbb{R}^n}(v, v) \leq (1 - \delta) g_{\mathbb{R}^m}(v, v)$ for some $\delta \in (0, 1]$

Idea: Set $s := g_{\mathbb{R}^m} - f^* g_{\mathbb{R}^n}$ and show $s \geq \varepsilon_1 g$

Lemma

Let $f : \mathbb{R}^m \to \mathbb{R}^n$ have bounded geometry and assume $s \geq \varepsilon_1 g$ for $\varepsilon_1 > 0$ at $t = 0$. Then $s \geq \varepsilon_1 g$ for $t \in [0, T)$.

Corollary

Any $F_t(\mathbb{R}^m)$ is the graph of a map $f_t : \mathbb{R}^m \to \mathbb{R}^n$.

Idea of the proof:

- Modify tensor s s.t. only a compact set needs to be considered.
- $\rightsquigarrow$ Usual (for compact domain) maximum principle applies!

[Proof of the Lemma]

1. Rewrite the condition:

$$f \text{ length-decreasing} \quad \Leftrightarrow \quad 0 \le \mathrm{g}_{\mathbb{R}^m}(v,v)_{|p} - f^*\mathrm{g}_{\mathbb{R}^n}(v,v)_{|p}$$

2. $\mathrm{s}_{\mathbb{R}^m\times\mathbb{R}^n} := \pi_M^*\mathrm{g}_M - \pi_N^*\mathrm{g}_N$. Then:

$$\mathrm{g}_{\mathbb{R}^m}(v,v)_{|p} - f^*\mathrm{g}_{\mathbb{R}^n}(v,v)_{|p} = F^*\,\mathrm{s}_{\mathbb{R}^m\times\mathbb{R}^n}(v,v)_{|p} =: \mathrm{s}(v,v)_{|p}$$

[Pullback of constant tensor]

3. Show $\mathrm{s} \ge \varepsilon\mathrm{g}$:
   1. Set $\psi_{|(x,t)} := \mathrm{e}^{\sigma t}\phi_R(x)\mathrm{s}_{|(x,t)} - \varepsilon\mathrm{g}_{|(x,t)}$,
      where $\phi_R(x) := 1 + \|x\|^2_{\mathbb{R}^m}/R^2$
   2. Assume $\mathrm{s} \ge \frac{\varepsilon}{2}\mathrm{g}$. [will be removed]
      $\Rightarrow \psi > 0$ outside some compact $K \subset \mathbb{R}^m$ for all $t \in [0,T)$.
   3. Inside $K \times [0,T']$ for any $T' \in [0,T)$, at a minimum at (x_0,t_0) we have

$$(\nabla_{\partial_t}\psi)(v,v) \le 0 \qquad \text{and} \qquad (\Delta\psi)(v,v) \ge 0.$$

   4. By calculation, it is $(\nabla_{\partial_t}\psi)(v,v) - (\Delta\psi)(v,v) > 0$ for any $R \ge R_0 > 0$ and a (chosen, but fixed) R_0. ⚡
   5. The claim follows by letting $R\to\infty$, $\sigma\to 0$ and $T'\to T$. □

Decay Estimate for $\|\vec{H}\|^2$

Idea:

Consider restriction of $\mathrm{s}_{\mathbb{R}^m \times \mathbb{R}^n}$ onto normal bundle. Compare with $\vec{H} \otimes \vec{H}$.

Normal Bundle Geometry

Fibers:

$$T_p^{\perp}\mathbb{R}^m := \{\xi \in T_{F(p)}(\mathbb{R}^m \times \mathbb{R}^n) : \langle \xi, v\rangle = 0\,, \forall v \in \mathrm{d}F(T_p\mathbb{R}^m)\}$$

Projection onto normal bundle:

$$\mathrm{pr}^{\perp} : F^* T(\mathbb{R}^m \times \mathbb{R}^n) \to T^{\perp}\mathbb{R}^m\,, \qquad \mathrm{pr}^{\perp}(v) := v - \sum_{k=1}^{m} \langle v, \mathrm{d}F(e_k)\rangle\, \mathrm{d}F(e_k)$$

[$\{e_1, \ldots, e_m\}$ local g-ON-frame for $T\mathbb{R}^m$]

Normal connection:

$$\nabla_v^{\perp}\xi := \mathrm{pr}^{\perp}\left(\nabla^{\mathrm{g}_{\mathbb{R}^m \times \mathbb{R}^n}}_{\mathrm{d}F(v)}\xi\right)\,, \qquad v \in \Gamma(T\mathbb{R}^m)\,,\ \xi \in \Gamma(T^{\perp}\mathbb{R}^m)$$

Definition: $$\vartheta(\xi, \eta) := \langle \vec{H}, \mathrm{pr}^\perp(\xi) \rangle \langle \vec{H}, \mathrm{pr}^\perp(\eta) \rangle$$

Idea for $M = N$ compact (Smoczyk, 2004):

Use isomorphism $J : TM \to T^\perp M$ and consider

$$\widetilde{\phi}(v, w) := \mathrm{s}(v, w) - \varepsilon_2 \vartheta(J\,\mathrm{d}F(v), J\,\mathrm{d}F(w)).$$

Show $\widetilde{\phi} \geq 0$.

Restriction of $\mathrm{s}_{\mathbb{R}^m \times \mathbb{R}^n}$ to normal bundle:

$$\mathrm{s}^\perp(\xi, \eta) := \mathrm{s}_{\mathbb{R}^m \times \mathbb{R}^n}(\mathrm{pr}^\perp(\xi), \mathrm{pr}^\perp(\eta))$$

with $\xi, \eta \in \Gamma(F^* T(\mathbb{R}^m \times \mathbb{R}^n))$.

Observation: $$\underbrace{\mathrm{s} \geq 0}_{\text{length-decr. map}} \quad \Rightarrow \quad \mathrm{s}^\perp \leq 0$$

In the case at hand:

Consider

$$\phi(\xi, \eta) := -\mathrm{s}^\perp(\xi, \eta) - \varepsilon_2 t \vartheta(\xi, \eta).$$

Show $\phi \geq 0$.

Decay Estimate for $\|\vec{H}\|^2$

Lemma

Let $F(x,t)$ be a smooth solution to the mean curvature flow with bounded geometry and suppose $\mathrm{s} - \varepsilon_1 \mathrm{g} \geq 0$ for some $\varepsilon_1 > 0$ at $t = 0$. Then there exists a constant $\varepsilon_2 > 0$ depending only on ε_1 and the dimension $m = \dim \mathbb{R}^m$, such that

$$\phi = -\mathrm{s}^{\perp} - \varepsilon_2 t \vartheta \geq 0$$

on $\mathbb{R}^m \times [0, T)$.

Corollary

Under the assumptions of the lemma, it is

$$t\|\vec{H}\|^2 \leq C$$

on $\mathbb{R}^m \times [0, T)$ for some constant $C \geq 0$ depending only on ε_1 and the dimensions $\dim \mathbb{R}^m$ and $\dim \mathbb{R}^n$.

Higher-Order Estimates I

Lemma

Let $F(x) = (x, f(x,t))$ be a smooth solution of the non-parametric mean curvature flow equation

$$\frac{\partial f}{\partial t}(x,t) = \sum_{i,j=1}^{m} \widetilde{\mathrm{g}}^{ij}\,\frac{\partial^2 f}{\partial x^i \partial x^j}$$

satisfying the bounded geometry condition. Suppose that

$$\|\mathrm{D}f\| \le C_1 \quad \text{and} \quad \|\mathrm{D}^2 f\| \le C_2$$

on $\mathbb{R}^m \times [0,T)$ for some $C_1, C_2 \ge 0$. Then for every $l \ge 3$ there is a constant C_l, such that

$$\sup_{x \in \mathbb{R}^m} \|\mathrm{D}^l f(x,t)\|^2 \le C_l$$

for all $t \in [0,T)$.

Proof relies on blow-ups.

Higher-Order Estimates II

Lemma

Let $F(x) = (x, f(x,t))$ be a smooth solution of the non-parametric mean curvature flow equation

$$\frac{\partial f}{\partial t}(x,t) = \sum_{i,j=1}^{m} \tilde{g}^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j}$$

satisfying the bounded geometry condition. Suppose that

$$f^* g_{\mathbb{R}^n} \leq (1-\delta) g_{\mathbb{R}^m} \quad \text{and} \quad \|\vec{H}\| \leq C$$

on $\mathbb{R}^m \times [0,T)$ for some $\delta \in (0,1]$ and $C \geq 0$. Then for every $l \geq 1$ there is a constant C_l, such that

$$\sup_{x \in \mathbb{R}^m} \|D^l f(x,t)\|^2 \leq C_l$$

for all $t \in [0,T)$.

Height Estimate

Lemma

Suppose f is a smooth solution to the non-parametric mean curvature flow equation

$$\frac{\partial f}{\partial t}(x,t) = \sum_{i,j=1}^{m} \tilde{g}^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j}$$

on a time interval $[0,T)$ that satisfies the bounded geometry condition. Then

$$\sup_{x\in\mathbb{R}^m} \|f(x,t)\|^2 \leq \sup_{x\in\mathbb{R}^m} \|f(x,0)\|^2$$

holds for all $t \in [0,T']$, where $T' \in [0,T)$ is arbitrary.

Proof:
Calculate

$$\left(\frac{\partial}{\partial t} - \sum_{i,j=1}^{m} \tilde{g}^{ij} \partial^2_{ij}\right) \|f\|^2 \leq 0$$

and apply a non-compact maximum principle.

Approximation of Initial Data

Set

$$K(x,y,t) := \frac{1}{(4\pi t)^{m/2}} \exp\left(-\frac{\|x-y\|^2_{\mathbb{R}^m}}{4t}\right)$$

and

$$f_0^k : \mathbb{R}^m \to \mathbb{R}^n, \qquad f_0^k(x) := \int_{\mathbb{R}^m} f_0(y) K\left(x, y, \frac{1}{k}\right) \mathrm{d}y.$$

If f_0 is Lipschitz continuous and strictly length-decreasing, then ...

1. $f_0^k \xrightarrow{k\to\infty} f_0$ in $\mathscr{C}^{0+\alpha}\big(B(0,R)\big)$ for any $R \geq 0$ and $0 < \alpha < 1$,
2. $\|\mathrm{d}f_0^k(v)\|_{\mathrm{g}_{\mathbb{R}^n}} \leq (1-\delta)\|v\|_{\mathrm{g}_{\mathbb{R}^m}}$ for any k,
3. $\sup_{\mathbb{R}^m} \|\mathrm{D}^l f_0^k\| < \infty$ for every $l \geq 2$ and k.

Higher-Order Estimates III

Lemma

Consider the sequence $\{f_0^k(x)\}$, *where* $f_0 : \mathbb{R}^m \to \mathbb{R}^n$ *is Lipschitz continuous and strictly length-decreasing. Then, for each* k, *the non-parametric mean curvature flow equation has a smooth solution* $f^k(x,t)$ *on* $\mathbb{R}^m \times [0,\infty)$ *with initial condition* $f_0^k(x)$, *such that*

1. $\| \mathrm{d}f^k(v)\|_{g_{\mathbb{R}^n}} \leq (1-\delta)\|v\|_{g_{\mathbb{R}^m}}$,
2. *for all* $l \geq 2$ *we have the estimate*

$$t^{l-1} \sup_{x\in\mathbb{R}^m} \|\mathrm{D}^l f^k(x,t)\|^2 \leq C_{l,\delta}$$

for some constant $C_{l,\delta} \geq 0$ *depending only on* l *and* δ.

Area-Decreasing Maps

Definition (Area-decreasing maps)

A map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is strictly area-decreasing iff

$$\| \mathrm{d}f(v) \wedge \mathrm{d}f(w)\|_{\Lambda^2 T\mathbb{R}^2} \leq (1-\delta)\|v \wedge w\|_{\Lambda^2 T\mathbb{R}^2}$$

for some (fixed) $\delta \in (0,1]$ and all $v, w \in \Gamma(T\mathbb{R}^2)$.

Example

- Let $B \in \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$ with $|\det(B)| < 1-\delta$, $b \in \mathbb{R}^2$.
- $f(x) := Bx + b$ is strictly area-decreasing

Theorem (FL, arXiv:1608.05394)

Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth strictly area-decreasing function satisfying $\|\mathrm{d}f(v)\| \leq c\|v\|$. Then the mean curvature flow with initial condition $F(x) := (x, f(x))$ has a long-time smooth solution for all $t > 0$. Further:

1. *Along the flow, the evolving surface stays the graph of a strictly area-decreasing map $f_t : \mathbb{R}^2 \to \mathbb{R}^2$ for all $t > 0$.*
2. *The mean curvature vector of the graph satisfies the estimate $t\|\vec{H}\|^2 \leq C$ for some $C \geq 0$.*
3. *All spatial derivatives of f_t of order $k \geq 2$ satisfy the estimate*

$$t^{k-1} \sup_{x\in\mathbb{R}^2} \|\mathrm{D}^k f_t(x)\|^2 \leq C_{k,\delta} \quad \text{for all} \quad k \geq 2$$

and for some constants $C_{k,\delta} \geq 0$ depending only on k and δ. Moreover,

$$\sup_{x\in\mathbb{R}^2} \|f_t(x)\|^2 \leq \sup_{x\in\mathbb{R}^2} \|f(x)\|^2$$

for all $t > 0$.

If in addition f satisfies $\|f(x)\| \to 0$ as $\|x\| \to \infty$, then $\|f_t(x)\| \to 0$ smoothly on compact subsets of $\mathbb{R}^2$ as $t \to \infty$.